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A TREATISE

ON THE

RESISTANCE OF MATERIALS,

And an Appendix

ON THE

PRESERVATION OF TIMBER.

BY

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PREFACE.

THE subject which forms the title of this work is inexhaustible. Volumes have been published containing the results of experiments, and yet experimental investigations, especially in regard to iron, were probably never more numerous than at the present time. The infinitely varied character of the materials, and of the great variety of conditions under which they are used, renders it impossible for a limited number of experiments to cover the whole ground. The most refined analysis has been brought to bear upon the subject, and yet many problems, which at first appear to be comparatively simple, remain unsolved. For instance, no theory of the rupture of a simple beam has yet been proposed which fully satisfies the critical experimenter.

Numerous theories have been proposed from time to time in regard to the resistance of materials under strain, but none are *universally* satisfactory. I do not agree with Barlow's theory of rupture involving his "Resistance to Flexure," and hence I have put all references to it in fine print, except the statement of its principles.

The general plan and scope of this edition are essentially the same as the former one. I have, however, omitted some matter which appeared to be unimportant, and have added considerable new matter which, I trust, will add to the scientific value of the work.

I have given considerable prominence to the subject of shearing stresses and strains. Shearing strains are somewhat analogous to the *flowing* of the particles over each other. As our knowledge of the subject becomes more critical, this branch of it becomes more important.

I have given a new formula for the deflection of a beam (Equation (219*a*)), but I have not sufficient data at hand of the proper kind to test its accuracy in practice. It will doubtless be tested, sooner or later, when its possible accuracy will be determined.

I have sought to present the subject in such light as to impress upon the mind of the student that he is learning only the rudiments, whilst a large field remains for time to explore.

It is with pleasure that I here acknowledge my indebtedness to Professor W. A. Norton, of New Haven, Conn., and to my colleague, Professor R. H. Thurston, of the Stevens' Institute of Technology, for valuable and original matter.

DE VOLSON WOOD.

HOBOKEN, February 27th, 1875.

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ON

THE RESISTANCE OF MATERIALS.

INTRODUCTION.

1. IN PROPORTIONING ANY MECHANICAL STRUCTURE, there are at least two general problems to be considered:—

1st. The nature and magnitude of the forces which are to be applied to the structure, such as moving loads, dead weights, force of the wind, etc.; and,

2d. The proper distribution and magnitude of the parts which are to compose the structures, so as to successfully resist the applied forces.

The former of these problems may be solved without any reference to the latter, as the structure may be considered as composed of rigid right lines. The latter depends principally upon the mechanical properties of the materials which compose the structure, such as their strength, stiffness, and elasticity, under various circumstances.

The mechanical properties of the principal materials—wood, stone, and iron*—have been determined with great care and expense by different experimenters, both in this and foreign countries, to which reference will hereafter be made.

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* The properties of mortars have been thoroughly discussed by Gen. Q. A. Gilmore in his work on *Limes, Mortars, and Cements.* 1862.

2. DEFINITIONS OF TERMS.

STRESSES are the forces which are applied to bodies to bring into action their elastic and cohesive properties. These forces cause alterations of the forms of the bodies upon which they act.

STRAIN is a name given to the kind of alterations produced by the *stresses*. The distinction between stress and strain is not always observed; one being used for the other. One of the definitions given by lexicographers for *stress*, is *strain*; and inasmuch as the kind of distortion at once calls to mind the manner in which the force acts, it is not essential for our purpose that the distinction should always be made.

A TENSILE STRESS, or *Pull*, is a force which tends to elongate a piece, and produces a strain of extension, or *tensile strain*.

A COMPRESSIVE STRESS, or *Push*, tends to shorten the piece, and produces a *compressive strain*.

TRANSVERE STRESS acts transversely to the piece, tending to bend it, and produces a *bending strain*. But as a compressive stress sometimes causes bending, we call the former a *transverse strain*, for it thus indicates the character of the stress which produces it. *Beams* are generally subjected to transverse strains.

TORSIVE STRESS causes a twisting of the body by acting tangentially, and produces a *torsive strain*.

LONGITUDINAL SHEARING STRESS, sometimes called a *detrusive strain*, acts longitudinally in a fibrous body, tending to draw one part of a solid substance over another part of it; as, for instrance, in attempting to draw the piece *A B*, Fig. 1, which has a shoulder, through the mortise *C*, the part forming the shoulder will be forced longitudinally off from the body of the piece, so that the remaining part may be drawn through. (See also Fig. 31.)

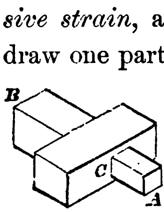

FIG. 1.

TRANSVERSE SHEARING STRESS is a force which acts transversely, tending to force one part of a solid body over the adjacent part. It acts like a pair of shears. It is the stress which would break a tenon from the body of a beam, by acting perpendicular to the side of the beam and close to the tenon. It is the stress which shears large bars of iron transversely, so often seen in machine-shops. The applied and resisting forces act in parallel planes, which are very near each other.

SPLITTING STRESS, as when the forces act normally like a wedge, tending to split the piece.

3. THE EFFECT OF THESE STRESSES IS TWOFOLD:—1st. Within certain limits they only produce change of form; and, 2d, if they are sufficiently great they will produce rupture, or separation of the parts; and these two conditions give rise to two general problems under the resistance of materials, the former of which we shall call the problem of ELASTIC RESISTANCE; the latter, ULTIMATE RESISTANCE, or RESISTANCE TO RUPTURE.

4. GENERAL PRINCIPLES OF ELASTIC RESISTANCES.—To determine the laws of elasticity we must resort to experiment. Bars or rods of different materials have been subjected to different strains, and their effects carefully noted.

From such experiments, made on a great variety of materials, and with apparatus which enabled the experimenter to observe very minute changes, it has been found that, whatever be the physical structure of the materials, whether fibrous or granular, they possess certain general properties, among which are the following:—

1st. That all bodies are elastic, and within very small limits they may be considered perfectly elastic; *i. e.*, if the particles of a body be displaced any amount within these limits they will, when the displacing force is removed, return to the same position in the mass that they occupied before the displacement. This limit is called *the limit of perfect elasticity.**

* Mr. Hodgkinson made some experiments to prove that all bodies are non-elastic. (See *Civil Eng. and Arch. Jour.*, vol. vi., p. 354.) He found that the limits of perfect elasticity were exceedingly small, and inferred that if our

2d. The amount of displacement within the elastic limit is directly proportional to the force which produces it. It follows from this, that in any prismatic bar the force which produces compression or extension, divided by the amount of extension or compression, will be a constant quantity.

3d. If the displacement be carried a little beyond this limit the particles will not return to their former position when the displacing force is removed, but a part or all of the displacement will be permanent. This Mr. Hodgkinson called a *set*, a term which is now used by all writers upon this subject.

4th. The amount of displacement is not exactly, but nearly, proportional to the applied force considerably beyond the elastic limit.

5th. Great strains, producing great sets, impair the elasticity.

5. COEFFICIENT (OR MODULUS*) OF ELASTICITY.

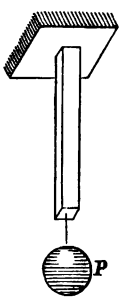

FIG. 2.

If a prismatic bar, whose section and length are unity, be compressed or elongated any amount within the elastic limit, the quotient obtained by dividing the force which produces the displacement by the amount of compression or extension is called the COEFFICIENT OF ELASTICITY. This we call E.

Let K = section of a prismatic bar (See Fig. 2),
l = its length,

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powers of observation were perfect in kind and infinite in degree, we should find that no body was perfectly elastic even for the smallest amount of displacement. And although more recent experiments have *indicated* the same result in cast-iron, yet the most delicate experiments have failed to thoroughly establish it. I have, therefore, accepted the principle of perfect elasticity, which, for the purposes of this work, is practically, if not theoretically, correct. It does not appear from Mr. Hodgkinson's report how soon the effect was observed after the strain was removed. If he had allowed considerable time the *set* might have disappeared, as it is evident that it takes time for the displaced particles to return to their original position.

* The terms *coefficient* and *modulus* are used indiscriminately for the constants which enter equations in the discussion of physical problems, and are sometimes called *physical constants*. The *modulus* of elasticity, as used by most writers on Analytical Mechanics, is the ratio of the force of restitution to

and λ = the elongation or compression caused by a force, P, which is applied longitudinally. Then

$\frac{P}{K}$ = force on a unit of section, and

$\frac{\lambda}{l}$ = the elongation or compression for a unit of length.

Hence, from the definition given above, we have

$$E = \frac{P}{K} \div \frac{\lambda}{l} = \frac{Pl}{K\lambda} \dots\dots\dots\dots\dots\dots (1)$$

From this equation E may be easily found. It will hereafter be shown that the coefficent is not exactly, but is nearly the same for compression as for tension.

For values of E, see Appendix III.

6. PROOFS OF THE LAWS GIVEN IN ARTICLE FOUR.—Article 5 has preceded these proofs, so as to show how the results of experiments may be reduced by equation (1). The 1st and 2d laws seem first to have been proved by S. Gravesend, since which they have been confirmed by numerous experimenters. One of the most extensive and reliable series of experiments upon various substances for engineering purposes is given in "The Report of Her Majesty's Commissioners, made under the direction of Mr. Eaton Hodgkinson." The results of his experiments are published in the Reports of the British Association, and in the 5th volume of the Proceedings of the Manchester Literary and Philosophical Society, from which extracts have been made and to which we shall have occasion to refer. The experiments were made not only to prove these laws but several others, principally relating to transverse strength.

Barlow made many experiments, the results of which are given in his valuable work on the "Strength of Materials."

that of compression. It relates to the impact of bodies, and, as thus defined, depends upon the set. But the *coefficient* of elasticity depends neither upon impact nor set. Another term should therefore be used, or else a distinction should be made between the terms *coefficient* and *modulus*, so that the former shall apply to small displacements, and the latter to the relative force of restitution. For this reason I have used the former in this work, and avoided the latter when applied to elasticity.

The series of experiments on iron which had been commenced and so ably conducted by Mr. Hodgkinson were continued by Mr. Fairbairn. The latter confined his experiments mostly to transverse strength, the results of which are given in his valuable work on "Cast and Wrought Iron." A valuable set of experiments has been made in France at "le Conservatoire des Arts et Métiers." *

In this country several very valuable sets of experiments have been made, among the most important of which are the experiments of the Sub-Committee of the Franklin Institute, the results of which are published in the 19th and 20th volumes of the Journal of that Society, commencing on the 73d page of the former volume. The experiments were made upon boiler iron, but they developed many properties common to all wrought iron. They were conducted with great care and scientific skill. The report gives a description of the testing machine; the manner of determining its friction and elasticity; the modifications for use in high temperature; the manner of determining the latent and specific heats of iron; and the strength of different metals under a variety of circumstances.

Another very valuable set of experiments was made by Captain T. J. Rodman and Major W. Wade, upon "Metals for Cannon, under the direction of the United States Ordnance Department," and published by order of the Secretary of War.

Numerous other experiments of a limited character have been made, too many of which have been lost to science because they were not reported to scientific journals, and many others were of too rude a character to be very valuable

The results of these experiments will form the basis of our theories and analysis.

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* See "*Morin's Résistance des Matériaux*," p. 126.

CHAPTER I.

TENSION.

7. ELASTIC RESISTANCE.—We will first consider the elastic resistance due to tension, or, as it is sometimes called, a pull, or elongating force.

EXPERIMENTS ON WROUGHT IRON.

Experiments for determining the total elongation and permanent elongation produced by different weights acting by extension on a tie of wrought iron of the best quality, by Eaton Hodgkinson.

Weight in kilogrammes per square centimetre. P.	Elongation per metre of length. Total. λ.	Elongation per metre of length. Permanent.	Coefficient of elasticity per square metre E.
Kil.	M.	Mill.	Kil.
187.429	0.000082117	....	22 824 500 000
374.930	0.000185261	....	20 216 200 000
562.406	0.000283704	0.00254	19 824 100 000
749.456	0.000379467	0.0033894	19 704 000 000
937.430	0.000475113	0.0042398	19 729 909 000
1124.813	0.000570792	0.00508	19 706 000 000
1312.283	0.000665647	9.0067705	19 714 600 000
1499.720	0.000760311	0.0100879	19 320 300 000
1687.219	0.000873265	0.0330283	19 320 700 000
1874.645	0.001012911	0.0829955	18 398 100 000
2063.580	0.001283361	0.2616950	16 079 200 000
2249.627	0.002227205	....	.. ..
2403.653	0.004287185	3.0709900	5 606 590 000
2624.564	0.009156490	8.4690700	2 866 380 000
....	0.009950970	8.5748700	.. ..
2812.033	0.010492805	9.1023600	2 681 520 000
Repeated after 1 hour.	0.011750313	....	.. ..
" " 2 "	0.011858889	....	.. ..
" " 3 "	0.011933837	....	.. ..
" " 4 "	0.011942168	....	.. ..
" " 5 "	0.011958835	....	.. ..
" " 6 "	0.011967149	....	.. ..
" " 7 "	0.012027114	....	.. ..
" " 8 "	0.012027014	....	.. ..

EXPERIMENTS ON WROUGHT IRON.—*Continued.*

Weight in kilogrammes per square centimetre. *P.*	Elongation per metre of length.		Coefficient of elasticity per square metre. *E.*
	Total. λ.	Permanent.	
Kil.	M.	Mill.	Kil.
Repeated after 9 hours.	0.012027114	....	.. ..
" " 10 "	0.012027114	....	.. ..
2999.500	0.017888263	16.5145	1 676 820 000
2999.500	0.019478898	....	.. ..
....	0.01984831	18.4212	.. ..
....	0.02022006	18.8886	.. ..
3186.978	0.02148590	19.7954	1 483 290 000
....	0.02169401	....	.. ..
....	0.02170242	....	.. ..
....	0.02170242	22.0119	.. ..
3374.440	0.02477441	22.7087	1 362 020 000
....	0.02514184	....	.. ..
....	0.02522512	....	.. ..
3561.900	0.03493542	32.8201	1 019 580 000
....	0.03519357	....	.. ..
....	0.03520190	....	.. ..
3745.361	....	....	.. ..

This table is given in French units because it was more convenient.*

8. The results of these experiments may be represented graphically by taking, as in Fig. 3, the total elongations or the permanent elongations for abscissas and the weights for ordinates.

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* To reduce the French measures to English we have the following relations:—

LINEAR MEASURE.

3.2808992 feet = 1 metre.
0.328089 feet = 1 centimetre.
0.0032808 feet = 1 millimetre.
0.0393696 in. = 1 millimetre.

WEIGHT.

2.20462 lbs. avoird. = 1 kilogramme.
1422.28 lbs. pr. sq. in. = 1 kilog. to the sq. millimetre.
0.00142228 lbs. sq. in. = 1 kilog. pr. sq. metre.

Hence to reduce the above quantities to English units, multiply the numbers in the first column by 14.2228 to reduce them to pounds avoirdupois per square inch; those in the second column by 3.28089 + to reduce them to feet; the third by 0.03936 + to reduce them to inches; and the fourth by 0.00142228 to reduce them to pounds per square inch.

When the construction is made on a large scale it makes the results of the experiments very evident.

An examination of Fig. 3 shows:—

1st. That to a load of 1499.72 kil. pr. square centimetre, the total elongations are practically proportional to the loads;

2d. That within the same limits the permanent elongations are nearly proportional to the loads, and that they are exceedingly small;

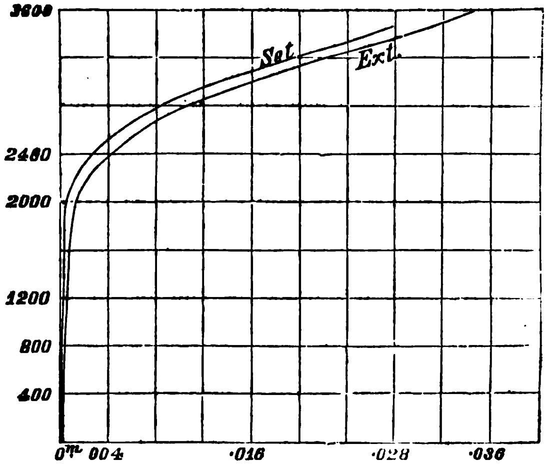

FIG. 3.

3d. That beyond the load of 14.997 kil. to 22.00 kil. per square millimetre, the total and permanent elongations increase very rapidly and more than proportional to the loads;

4th. That near and beyond 22.49 kil. per square millimetre, the total elongations become sensibly proportional to the loads, but in a much greater ratio than that which corresponds to small loads. For the loads near rupture the elongations are a little inferior to that indicated by the new proportion.

5th. Beyond 14.99 kil. per square millimetre, the permanent elongations increase much more rapidly than the total elongations. We also observe that the permanent elongations increase with the duration of the load, although very slowly. The latter property will be more particularly noticed hereafter.

6th. Finally, the values $\frac{P}{\lambda}$ of the loads per square metre to the elongation per metre, and which is called the *coefficient of elasticity*, is sensibly constant when the elongations are nearly proportional to the loads; and that the mean value is

$$E = 19{,}816{,}440{,}000 \text{ kil. per square metre;}$$
$$= 28{,}283{,}000 \text{ lbs. per square inch.}$$

The first value of E, in the table, is much larger, and may have resulted from an erroneous measurement of the exceedingly small total elongation. From the experiments made on another bar, Hodgkinson found

$$E = 19{,}359{,}458{,}500 \text{ kil. per sq. metre};$$
$$= 27{,}700{,}000 \text{ pounds pr. square inch};$$

which is but little less than the preceding.

Mr. Hodgkinson infers from these experiments that the smallest strains cause a permanent elongation. But Morin forcibly remarks * that none of these experimenters appear to have verified whether time, after the strains are removed, will not cause the permanent elongations to disappear. Also that the deflections of the machine cannot be wholly eliminated, and hence appear to increase the true result. In practice such small permanent elongations may be omitted.

The preceding example has, for a long time, been given to show the law of relation between the applied force and the total and permanent elongations; but we should not expect to find exactly the same results for all kinds of iron. Even wrought iron has such a variety of qualities, depending upon the ore of which it is made, and the process of manufacture, that it cannot be expected that the above results will always be applicable to it. Only a wide range of experiments will determine how far they may generally be relied upon.

It is found, however, that the GENERAL RESULTS of extension, of set, of increased elongation with the duration of the stress within certain limits, and of the increase of set with the increase of load, are true of all kinds of iron.

EXPERIMENTS UPON CAST IRON.

9. THE FOLLOWING EXPERIMENTS UPON CAST-IRON show that the numerical relation between the applied force and the extension is somewhat different from the preceding. The experiments were made under the supervision of Captain T. J. Rodman :—†

" The specimens had collars left on them at a distance of thirty-five inches

* Morin's *Résistance des Matériaux*, p. 10.

† *Experiments on Metals for Cannon*, by Capt. T. J. Rodman, p. 157.

For a full description of the testing apparatus, with diagrams, see Major Wade's Report on the *Strength of Materials for Cannon*, pp. 305–315. The machine consists principally of a very substantial frame and levers resting on knife edges.

apart, the space between the collars being accurately turned throughout to a uniform diameter.

"The space between the collars was surrounded by a cast-iron sheath, eight-tenths of an inch less in length than the distance between the collars; it was put on in halves and held in position by bands, and was of sufficient interior diameter to move freely on the specimen.

"When in position, the lower end of the sheath rested on the lower collar of the specimen, the space between its upper end and the upper collar being supplied with and accurately measured by a graduated scale tapered 0.01 of an inch to one inch.

"The upper end of the sheath was mounted with a vernier, and the scale was graduated to the tenth of an inch.

"This afforded means of measuring the changes of distance between the collars to the ten-thousandth part of an inch, and these readings divided by the distance between the collars gave the extension per inch in length as recorded in the following table:—

TABLE

Showing the extension and permanent set per inch in length caused by the under-mentioned weights, per square inch of section, acting upon a solid cylinder 35 *inches long and* 1.366 *inches diameter.* (*Cast at the West Point Foundry in* 1857.)

Weight per square inch of section.	Extension per inch of length.	Permanent set per inch in length.	Coefficient of elasticity.
P.	λ.		*E.*
lbs.	in.	in.	
1,000	0.0000611	0.	16,366.612
2,000	0.0000794	0.	25,189,168
3,000	0.0001089	0.	27,548,209
4,000	0.0001771	0.	22,586,674
5,000	0.0002129	0.	23,489,901
6,000	0.0002700	0.0000014	22,222,222
7,000	0.0003328	0.0000029	21,033,653
8,000	0.0003986	0.0000043	20,070,245
9,000	0.0004557	0.0000071	19,749,835
10,000	0.0005100	0.0000109	19,607,843
11,000	0.0005500	0.0000157	20,000,000
12,000	0.0006414	0.0000257	18,693,486
13,000	0.0007100	0.0000300	18,309,859
14,000	0.0007700	0.0000357	18,181,181
15,000	0.0008557	0.0000477	17,529,507
16,000	0.0009243	0.0000529	17,310,397
17,000	0.0010014	0.0000643	16,977,231
18,000	0.0010900	0.0001014	16,537,614
19,000	0.0012271	0.0001471	15,483,660
20,000	0.0013586	0.0002014	14,721,109
21,000	0.0015386	0.0002900	13,648,771
22,000	0.0017043	0.0003986	12,908,523
23,000	0.0019529	0.0005529	11,265,246
24,000	0.0022786	0.0007529	10,532,344
25,000	0.0026037	0.0010843	9,601,720
26,000	0.0032186		8,078,046

10. FIGURE 4 IS A GRAPHICAL REPRESENTATION OF THE ABOVE TABLE, constructed in the same way as Figure 3.

Experiments were made upon many other pieces, from which I have selected four, and called them *A*, *B*, *C*, and *D*, a graphical representation of which is shown in Figure 5. The right hand lines represent extensions, the left hand sets.

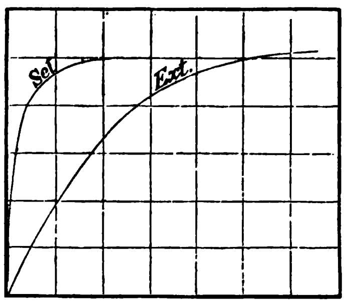

FIG. 4.

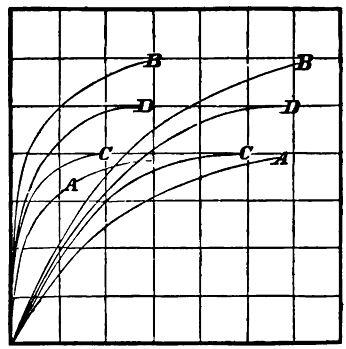

FIG. 5.

A was from an inner specimen of a Fort Pitt gun, No. 335, and the others from different cylinders which were cast for the purpose of testing the iron.

From these we observe:—

1st. That for small elongations the ratio of the stresses to the elongations is nearly constant.

2d. There does not appear to be a sudden change of the rate of increase, as in Mr. Hodgkinson's example, but the ratio gradually increases as the strains increase.

3d. The sets at first are invisible, but they increase rapidly as the strains approach the breaking limit.

It appears *paradoxical* that the first and second experiments in the preceding table should give a less coefficient than the third, but the same result was observed in several cases.

11. THE FOLLOWING TABLES ARE THE RESULTS OF SOME EXPERIMENTS MADE BY MR. HODGKINSON:—

Direct longitudinal extension of round rods of cast iron, fifty feet long.

NAME OF IRON.	No. of experiments.	Mean area of section.	Weights per square inch laid on, with their corresponding extensions and sets.			Mean breaking weight per square inch of section.	Mean ultimate extension.
			Weights. P.	Extension. λ.	Sets.		
			lbs.	in.	in.	lbs.	in.
Low Moor, No. 2......	2	1.058	2,117	0.0950	0.00345	16,408	1.085
			6,352	0.3115	0.0250		
			10,586	0.5640	0.4425		
			14,821	0.9147	0.12775		
Blaneavon Iron, No. 2..	2	1.0685	2,096	0.0942	0.00268	14,675	0.9325
			6,289	0.3065	0.01675		
			10,482	0.5770	0·0575		
			13,627	0.8370	0.11475		
Gartsherrie Iron, No. 2.	2	1.062	2,109	0.0922	0.001 +	16,951	1.167
			6,328	0.3117	0.01450		
			10,547	0.5862	0.0475		
			14,766	0.9452	0.11352		

In these experiments the ratio of the extensions is somewhat greater than that of the weights. The value of E, as computed for the first weights which are given, and the corresponding extensions, is a little more than 13,000,000 pounds per square inch.

Extension of cast-iron rods, ten feet long and one inch square.				Error in parts of the weight when it is computed from formula $P = 1161\lambda_e - 201905\lambda_e^2$.
Weights P.	Extensions. λ_e.	Sets.	$\frac{P}{\lambda_e}$.	
lbs.	in.	in.		
1053 77	.0090	.00022	117086	$-\frac{1}{42}$
1580.65	.0137	.000545	115131	$-\frac{1}{37}$
2167.54	.0186	.00107	113309	$-\frac{1}{119}$
3161.31	.0287	.00175	110150	$+\frac{1}{640}$
4215.08	.0391	.00265	107803	$+\frac{1}{257}$
5262.85	.0500	.00372	105377	$+\frac{1}{163}$
6322.62	.0613	.00517	103142	$+\frac{1}{177}$
7376.39	.0734	.00664	100496	$+\frac{1}{125}$
8430.16	.0859	.00844	98139	$+\frac{1}{155}$
9483.94	.0995	.01062	95316	$+\frac{1}{134}$
10537.71	.1136	.01306	92762	$+\frac{1}{121}$
11591.48	.1283	.01609	90347	$-\frac{1}{674}$
12645.25	.1448	.02097	87329	$-\frac{1}{195}$
13699.83	.1668		82133	$+\frac{1}{268}$
14793.10	.1859	.02410	79576	$-\frac{1}{80}$

Let P = the elongating force and

λ_e = the total elongation in inches due to P.

Then Hodgkinson found, from an examination of the table, that the empirical formula

$$P = 116117\lambda_e - 201905\lambda_e^2$$

represented the results more nearly than equation (1). This formula reduced to an equivalent one for l in inches (observing that the bar was 10 feet long), becomes

$$P = 13{,}934{,}000\frac{\lambda_e}{l} - 2{,}907{,}432{,}000\frac{\lambda_e^2}{l^2}$$

Although this equation gives the elongations for a greater range of strains than equation (1) for this particular case, yet the law represented by it is more complicated, and hence would make the discussions under it more difficult, without yielding any corresponding advantage. It is the equation of a parabola in which P is the abscissa and λ_e the ordinate.

We also see that when the elongations are very small, the quantity $\frac{\lambda_e^2}{l^2}$ will be very small, and the second term may be omitted in comparison with the first, in which case it will be reduced to equation (1). The coefficient in the first term is the coefficient of elasticity, hence it is nearly 14,000,000 lbs. for extension.

MALLEABLE IRON.

12. According to Barlow's experiments malleable iron may be elongated $\frac{1}{1000}$ of its length without endangering its elasticity.* To ascertain this, the strains were removed from time to time, and it was found that the index returned to zero for all strains less than 9 or 10 tons. The mean extension per ton (of 2,240 lbs.) per square inch, for four experiments, was 0.00009565 of its original length. Hence the mean value of the coefficient of elasticity is

$$E = \frac{P}{\lambda} = \frac{2240}{0.00009565} = 23{,}418{,}000 \text{ lbs.}$$

* *Journal Frank Inst.*, vol. xvi., 2d Series, p. 126.

ELASTICITY OF WOOD.

13. EXPERIMENTS BY MESSRS. CHEVANDIER AND WERTHEIM.—The following are some of the results of the recent experiments of Messrs. Chevandier and Wertheim on the resistance of wood. These experimenters have drawn the following principal conclusions:—

1. The density of wood appears to vary very little with age.
2. The coefficient of elasticity diminishes, on the contrary, beyond a certain age; it depends, likewise, upon the dryness and the exposure of the soil to the sun in which the trees have grown; thus the trees grown in the northern exposures, north-eastern, north-western, and in dry soils, have always so much the higher coefficient as these two conditions are united, whereas the trees grown in muddy soils present lower coefficients.
3. Age and exposure influence cohesion.
4. The coefficient of elasticity is affected by the soil in which the tree grows.
5. Trees cut in full sap, and those cut before the sap, have not presented any sensible differences in relation to elasticity.
6. The thickness of the woody layers of the wood appeared to have some influence on the value of the coefficient of elasticity only for fir, which was greater as the layers were thinner.
7. In wood there is not, properly speaking, any limit of elasticity, as every elongation produces a set.

It follows from this circumstance that there is no limit of elasticity for the woods experimented upon by Messrs. Chevandier and Wertheim; but, in order to make the results of their experiments agree with those of their predecessors, the authors have given for the value of the limit of elasticity the load under which it produces only a very small permanent elongation; the limit which they indicate in the following table for loads, under which the elasticity of wood commences to change, corresponds to a permanent elongation of 0.00005 of its original length.

TABLE CONTAINING THE MEAN RESULTS OF THE EXPERIMENTS OF MESSRS. CHEVANDIER AND WERTHEIM.

Species.	Density.	Coefficient of elasticity E referred to the square millimetre.	Limit of elasticity, or load per square millimetre, corresponding to that limit.	Cohesion, or load per square millimetre capable of producing rupture.	Val. of E in pounds per square inch.
		Kilogr.	Kilogr.	Kilogr.	
Locust	0.717	1261.9	3.188	7.93	The coefficient of elasticity varies from 1,000,000 lbs. to nearly 1,800,000 lbs. per square inch.
Fir	0.493	1113.2	2.153	4.18	
Yoke Elm	0.756	1085.3	1.282	2.99	
Birch	0.812	997.2	1.617	4.30	
Beech	0.823	980.4	2.317	3.57	
Oak from pedunculate acorn.	0.808	977.8	"	6.49	
" " sessile acorn	0.872	921.8	2.349	5.66	
White Pine	0.559	564.1	1.633	2.48	
Elm	0.723	1165.3	1.842	6.99	
Sycamore	0.692	1163.8	1.139	6.16	
Ash	0.697	1121.4	1.246	6.78	
Alder	0.601	1108.1	1.121	4.54	
Aspen	0.602	1075.9	1.035	7.20	
Maple	0.674	1021.4	1.068	3.58	
Poplar	0.477	517.2	1.007	1.97	

14. ELASTICITY OF WOOD, TANGENTIALLY AND RADIALLY.—The same observers have also determined the coefficient of elasticity and the cohesion of wood in the direction of the radius and in the direction of the tangent to the woody layers.

An examination of the following Table shows that the resistance in the direction of the radius is always greater than the resistance in the direction of the tangent to the woody layers; the relation between the coefficients of elasticity in the two cases varying nearly from 3 to 1.15.

MEAN RESULTS OF THE EXPERIMENTS OF MESSRS. CHEVANDIER AND WERTHEIM.

SPECIES.	IN THE DIRECTION OF RADIUS.		IN THE DIRECTION OF THE TANGENT TO THE LAYERS.	
	Coefficient of Elasticity, E, per square millimetre.	Cohesion, or load, per square millimetre, capable of producing rupture.	Coefficient of Elasticity, E, per square millimetre.	Cohesion, or load, per square millimetre, capable of producing rupture.
	Kilogr.	Kilogr.	Kilogr.	Kilogr.
Yoke Elm	208.4	1.007	103.4	0.608
Sycamore	134.9	0.522	80.5	0.610
Maple	157.1	0.716	72.7	0.371
Oak	188.3	0.582	129.8	0.406
Birch	81.1	0.823	155.2	1.063
Beech	269.7	0.885	159.3	0.752
Ash	111.3	0.218	102.0	0.408
Elm	121.6	0.345	63.4	0.366
Fir	94.5	0.220	34.1	0.297
Pine	97.7	0.256	28.6	0.196
Locust	170.3	"	152.2	1.231

The highest coefficient of elasticity in this table is for beech, and this is less than 400,000 pounds per square inch.

15. REMARK.—*The value of E*, which is used in practice, is not the coefficient of *perfect* elasticity, but it is that value which is nearly constant for small strains. In determining it, no account is made of the set. If the total elongations were proportional to the stresses which produce them, we would use the value of E found by them, even if the permanent equalled the total elongations. But in practice the permanent elongations will be small compared with the total for small stresses.

APPLICATIONS.

16. TO FIND THE ELONGATION OF A PRISMATIC BAR SUBJECTED TO A LONGITUDINAL STRAIN WHICH IS WITHIN THE ELASTIC LIMITS.

From (1) we have

$$\lambda = \frac{Pl}{EK} \quad \ldots\ldots\ldots\ldots \quad (2)$$

which is the required formula.

Also from (1) we have

$$P = \frac{\lambda}{l} EK \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

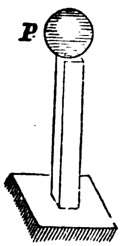

FIG. 6.

Equations (1), (2), and (3) are equally applicable to compressive strains, as will hereafter be shown. If in (3) we make $K = 1$ and $\lambda = l$ we shall have $P = E$; hence, the *coefficient of elasticity* may be defined to be *a force which will elongate a bar whose section is unity, to double its original length, provided the elasticity of the material does not change.* But there is no material, not even a perfectly elastic body—as air and other gases—whose *coefficient* of elasticity will not change for a perceptible change of volume. The material may not lose its elasticity, but equation (1) only measures it for small displacements. To illustrate further, let it be observed that, according to Mariotte's law, the volumes of a gas are inversely proportional to the compressive (or extensive) forces; double the force producing a compression to half the volume; four times the force, to one-fourth the volume, and so on, the compressions being a *fractional* part of the original volume; but in equation (2), λ is a linear quantity, so that if one pound produces an extension (or compression) of one inch, two pounds would produce an extension of two inches, and so on.

Examples.—1. If the coefficient of elasticity of iron be 25,000,000 lbs., what must be the section of an iron bar 60 feet long, so that a weight of 5,000 lbs. shall elongate it $\frac{1}{2}$ an inch.

From (1) we obtain $K = \frac{Pl}{E\lambda}$ which by substitution becomes

$$K = \frac{5,000.12 \times 60}{25,000,000 \times \frac{1}{2}} = 0.288 \text{ square inches.}$$

2. What weight will a brass wire sustain, whose diameter is 1 inch, coefficient of elasticity is 14,000,000 lbs., so as to elongate it $\frac{1}{800}$ of its length?

Ans. 13,744.5 lbs.

17.—REQUIRED THE ELONGATION (OR COMPRESSION) OF A PRISMATIC BAR WHEN ITS WEIGHT IS CONSIDERED.

Let l = the whole length of the bar before elongation or compression,
x = variable distance = Ab,
dx = bc = an element of length,
w = weight of a unit of length of the bar,
W = weight of the bar, and
P_1 = the weight sustained by the bar.

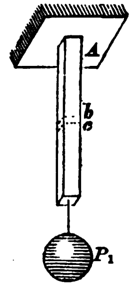

FIG. 7.

Then $(l - x)w + P_1 = P$ = the strain on any section, as bc.

Hence, from equation (2), we have

$$\lambda = \int_0^l \frac{P_1 + (l - x)w}{EK} dx = \frac{P_1 l + \frac{1}{2}wl^2}{EK} \qquad (4)$$

∴ the total length will become,

$$l \pm \lambda = \left[1 \pm \frac{P_1 + \frac{1}{2}wl}{EK}\right] l \qquad (5)$$

If $P_1 = 0$, $\lambda = \frac{wl_2}{2EK} = \frac{Wl}{2EK}$, or the total elongation is one-half of what it would be if a weight equal to the whole weight of the bar were concentrated at the lower end.

REQUIRED THE ELONGATION (OR COMPRESSION) OF A CONE IN A VERTICAL POSITION, CAUSED BY ITS OWN WEIGHT WHEN IT IS SUSPENDED AT ITS BASE (OR RESTS ON ITS BASE).

Take the origin at the apex before extension, Fig. 8, and

let K = any section,
K_0 = the upper section,
l = the length or altitude of the cone,
x = the length or altitude of any portion of the cone, and
δ = the weight of a unit of volume.

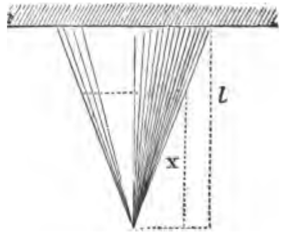

FIG. 8.

Then, because the bases of similar cones are as the squares of their altitudes, $K = K_0 \frac{x^2}{l^2}$

The volume of the cone whose altitude is x

$$=\int_0^x K dx = \int_0^x K_0 \frac{x^2}{l^2} dx = \tfrac{1}{3} K_0 \frac{x^3}{l^2},$$

and the weight of the same part

$$= \tfrac{1}{3} \delta K_0 \frac{x^3}{l^2}$$

$$\therefore \text{ (from equation (2)) } \lambda = \int_0^l \frac{\tfrac{1}{3} \frac{\delta K_0}{l^2} x^3}{E K_0 \frac{x^2}{l^2}} dx = \frac{1}{6} \frac{\delta l^2}{E}$$

from which it appears that the total elongation is independent of the transverse section, and varies as the square of the length.

18. THE WORK OF ELONGATION.—If P be the force which does the work, and x the space over which it works, then the general expression for the work is

$$U = \int_0^x P dx \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

To apply this to the elongation of a prism, substitute P from Eq. (3) in (6), and make $dx = d\lambda$, and we have

$$U = \int_0^\lambda \frac{EK\lambda}{l} d\lambda = \frac{EK\lambda^2}{2l} = \tfrac{1}{2} P\lambda \ldots\ldots (7)$$

which is the same result that we would have found by supposing that P was put on by increments, increasing the load gradually from zero to P.

Example.—If the coefficient of elasticity of wrought iron be 28,000,000 lbs., and is expanded 0.00000698 of its length for one degree F., how much work is done upon a prismatic bar whose section is one inch, and length 30 feet, by a change of 20 degrees of temperature?

Walls of buildings which were sprung outward have been drawn into an erect position by heating and cooling bars of iron. Several rods were passed through the building, and extending from wall to wall, were drawn tight by means of the nuts. Then a part of them were heated, thus elongating them, and the nuts tightened; after which they were allowed to cool, and the contraction which resulted drew the walls together. Then the other rods were treated in a similar manner, and so on alternately.

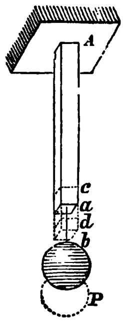

FIG. 9.

19. VERTICAL OSCILLATIONS.—If a bar Aa, Fig. 9, with a weight, P, suspended from its lower end, be pressed down by the hand, or by an additional weight from a to b, and the additional force be suddenly removed, the end of the bar on returning will not stop at a, but will move to some point above, as c, a distance $ac = ab$. From a principle in Mechanics, viz., that the *living force* equals twice the work, we are enabled to determine all the circumstances of the oscillation when the weight of the bar is neglected. The weight P elongates the bar so that its lower extremity is at a, at which point we will take the origin of co-ordinates.

Let $\lambda = ab =$ the elongation caused by the additional force,
$x = ad =$ any variable distance from the origin,
$v =$ the velocity at any point, as d, and
$M =$ the mass of the weight P.

If the weight of the rod be very small compared with P, the *vis viva* is

$$Mv^2 = \frac{P}{g} v^2 \text{ very nearly.}$$

The work for an elongation equal to λ, is by Eq. (7), $\frac{EK}{2l} \lambda^2$

The work for an elongation equal to x, is by Eq. (7), $\frac{EK}{2l} x^2$

$$\therefore \tfrac{1}{2} \frac{P}{g} v^2 = \frac{EK}{2l} (\lambda^2 - x^2), \text{ or } \frac{dx^2}{dt^2} = g \frac{EK}{Pl} (\lambda^2 - x^2) = v^2$$

$$\therefore t = \sqrt{\frac{Pl}{gEK}} \int_0^\lambda \frac{dx}{\sqrt{\lambda^2 - x^2}} = \left[\sqrt{\frac{Pl}{gEK}} \sin^{-1}\frac{x}{\lambda}\right]_0^\lambda = \frac{\pi}{2}\sqrt{\frac{Pl}{gEK}}$$

for half an oscillation; and the time for a whole oscillation is

$$T = \pi\sqrt{\frac{Pl}{gEK}} = \pi\sqrt{\frac{\lambda}{g}} \qquad (8)$$

hence the oscillations will be isochronous.

It is evident that by applying and removing the force at regular intervals, the amplitude of the oscillations may be increased and possibly produce rupture. In this way the Broughton suspension bridge was broken.*

As a second example, take the case in which P is applied suddenly to the end of the rod. It is evident that the total elongation will be greater than λ,—the permanent elongation. For the fundamental equation we may use another

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* Mr. E. Hodgkinson, in the 4th volume of the *Manchester Philosophical Transactions*, gives the circumstances of the failure, from this cause, of the suspension bridge at Broughton, near Manchester, England. And M. Navier, in his theory of suspension bridges (*Ponts Suspendus*, Paris, 1823), states that the duration of the oscillation of chain bridges may be nearly six seconds.

principle in Mechanics, which might have been used in the preceding problem, viz., that the mass multiplied by the acceleration equals the resultant moving force. The resisting force for an elongation x is $\frac{EKx}{l}$ (See Eq. (3)), and the moving force is P, whose mass $= \frac{P}{g}$; hence

$$M\frac{d^2x}{dt^2} = P - \frac{EK}{l}x;$$

$$\therefore \frac{dx^2}{dt^2} = \frac{g}{P}(2Px - \frac{EK}{l}x^2) = v^2$$

$$\therefore t = \sqrt{\frac{P}{g}}\int_0^x \frac{dx}{\sqrt{2Px - \frac{EKx^2}{l}}} = \sqrt{\frac{P}{g}\cdot\frac{l}{EK}}\text{ versin}^{-1}\frac{EKx}{Pl}$$

$$\text{If } x = \lambda,\ v = \sqrt{g\lambda},$$
$$x = 2\lambda,\ v = 0,$$
$$x = 0,\ v = 0.$$

Hence, the amplitude is twice the permanent elongation. If $x = 2\lambda$ we have $t = \pi\sqrt{\frac{Pl}{gEK}} = \pi\sqrt{\frac{\lambda}{g}}$. Investigations of this kind give rise to a division of the subject called *Resilience of Prisms.*

The investigations are interesting, but the results are of little use beyond those which have already been indicated. From the last problem we see that a weight suddenly applied produces twice the strain that it would if applied gradually.

As additional exercises for the student, I suggest the following: Suppose the weight be applied with an initial velocity. Suppose a weight P is attached to one end, and the weight P' is placed suddenly upon it; or it falls upon it. To find the velocity at any point in terms of t, — also λ in terms of t.

If a weight W is suspended at the end, and another weight W_1 falls from a height h, giving rise to a velocity v, we have for the common velocity of the bodies after impact, if both are non-elastic, $V = \frac{W_1 v}{W_1 + W}$, and the *vis viva* of both will be

$$MV^2 = \frac{W_1^2 v^2}{g(W_1 + W)}, \text{ which equals } \frac{EK}{l}\lambda^2, \text{ or twice the work.}$$

$$\therefore \lambda = \frac{W_1}{\sqrt{(W_1 + W)}}\sqrt{\frac{2hl}{EK}}.$$

This is only an approximate value, for the inertia of the wire is neglected.

20. VISCOSITY OF SOLIDS.—Experiments show that the principle of equal amplitudes, referred to in the preceding article, is not realized in practice. This is more easily observed in transverse vibrations. The amplitudes grow rapidly less from the first vibration, and the diminution cannot be fully accounted for by the external resistance of air. Professor Thompson of England has shown that there is an internal resistance which opposes motion among the particles of a body, and is similar to that resistance in fluids which opposes the movement of particles among themselves. He therefore called it *viscosity.** He proved:—

1st. That there was a certain internal resistance which he called Viscosity, and which is independent of the elastic properties of metals;

2d. That this force does not affect the co-efficient of elasticity.

The law between molecular friction and viscosity was not discovered.

The viscosity was always much increased at first by the increase of weight, but it gradually decreased, and after a few days became as small as if a lighter weight had been applied. Only one experiment was made to determine the effect of continual vibration; and in that the viscosity was very much increased by daily vibrations for a month.

This latter fact, if firmly established, will prove to be highly important; for it shows that materials which are subjected to constant vibrations, such as the materials of suspension bridges, have within themselves the property of resisting more and more strongly the tendency to elongate from vibration. Experiments will be given hereafter which tend to confirm this fact, when the vibrations are not too frequent or too severe.

But the true viscosity of solids has been fully proved by M. Tresca, a French physicist, who showed that when solids are subjected to a very great force, the amount of the force depending upon the nature of the material, that the particles in the immediate vicinity of pressure will *flow* over each other, so as to resemble the flowing of molasses, or tar, or other viscous fluids. By applying sufficient pressure solid bodies may be made to *flow* through holes in other bodies. Thus, true viscosity differs entirely in its character from the property recognized by Professor Thompson.

* *Civ. Eng. Jour.*, vol. 28, p. 322.

RESISTANCE TO RUPTURE BY TENSION.

21. MODULUS OF STRENGTH.—Many more experiments have been made to determine the ultimate resistance to rupture by tension than there have to determine the elastic resistance. In the earlier experiments the former was chiefly sought, and more recently all who experimented upon the latter also determined the former.

The force which is necessary to pull asunder a prismatic bar whose section is one square inch, when acting in the direction of the axis of the bar, is called the *modulus of strength.* This we call T. It expresses the *tenacity* of the material, and is sometimes called the absolute strength and sometimes *modulus of tenacity.*

22. FORMULA FOR THE MODULUS OF STRENGTH; *or the force necessary to break a prismatic bar, when acted upon by a tensile strain.*

Let K = the section of the bar in inches;
T = the modulus of tenacity; and
P = the required force.

It is proved by experiment that the resistance is proportional to the section; hence

$$P = TK \quad \text{(9)}$$

$$\therefore T = \frac{P}{K} \quad \text{(10)}$$

From (10) T may be found. In (10) if P is not the ultimate resistance of the bar, then will T be the strain on a unit of section.

From (9) we have

$$K = \frac{P}{T} \quad \text{(11)}$$

which will give the section.

The following are some of the values of T which have been found from experiment by the aid of Equation (10).

	Cohesive force or Tenacity in lbs. per square inch.
Ash (*English*)	17,000
Oak (*English*)	9,000 to 15,000
Pine (*pitch*)	10,500
Cast Iron *	14,800 to 16,900
Cast Iron (*Weisbach & Overman*)	20,000
Wrought Iron	50,000 to 65,000
Steel Wire	100,000 to 120,000
Bessemer Steel †	120,000 to 129,000
Bessemer Steel ‡	72,000 to 101,000
Bars of Crucible Steel §	70,000 to 134,000
Chrome Steel ‖	115,780 to 190,680

The most remarkable specimen of cast steel for tenacity which is on record was manufactured in Pittsburgh, Pa. It was tested at the Navy Yard at Washington, D. C., and was found to sustain 242,000 lbs. to the square inch! ¶

For other values see the Appendix.

23. *A vertical prismatic bar is fixed at its upper end, and a weight P_1 is suspended at the other; what must be the upper section at A, Fig.* **7**, *so as to resist n-times all the weight below it, the weight of the bar being considered?*

Let δ = the weight of a unit of volume of the bar, and the other notation as before.

Then $KT = nP_1 + n\delta Kl$.

$$\therefore \quad K = \frac{nP_1}{T - n\delta l} \qquad (12)$$

If $n = 1$, $K = \dfrac{P_1}{T - \delta l}$; and if $\delta l = T$, $K = \infty$, or no section is possible, and $l = \dfrac{T}{\delta}$ is the corresponding length of the bar.

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* *Hodgkinson*, Bridges. Weale, sup., p. 25.

† *Jour. Frank. Inst.* Vol. 84, p. 366.

‡ Also Experiments by Wm. Fairbairn, *Van Nostrand's Ec. En. Mag.*, Vol 1, p. 273. § Do. p. 1009.

‖ *Report*, Cat., J. B. Eads, C.E.

¶ *Am. R. R. Times* (Boston), Vol. 20, p. 206.

24. BAR OF UNIFORM STRENGTH. *Suppose a bar is fixed at its upper extremity, Fig.* **10**, *and a weight P_1 is suspended at its lower extremity; it is required to find the form of the bar so that the horizontal sections shall be proportional to the strains to which they are subjected—the weight of the bar being considered.*

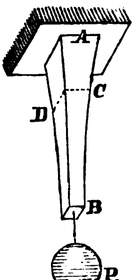


FIG. 10.

Let δ = weight of an unit of volume,

W = weight of the whole bar,

$K_0 = \frac{P_1}{T}$ = the section at B (Eq. (11)),

K_1 = the upper section,

K = variable section, and

x = variable distance from B upwards.

Also let the sections be similar:

Then $P = P_1 + \delta \int K\,dx$ = strain on any section, as $D\,C$. But TK is the ability to resist this strain:

$\therefore P_1 + \delta \int K dx = TK$. Differentiate this, and we have

$$\delta K\,dx = TdK$$

or $\frac{\delta}{T}\,dx = \frac{dK}{K}$ which by integrating gives

$$\frac{\delta}{T}\,x = Nap.\,log\,K + C \ldots\ldots\ldots\ldots \quad (12a)$$

But for $x = 0$, we have $K = K_0 \therefore C = -\,Nap.\,log\,K_0 = -\,Nap.\,log\,\frac{P_1}{T}$. Hence Eq. (12$a$) becomes $\frac{\delta}{T}\,x = Nap.\,log\,\frac{K}{K_0}$ or, passing to exponentials, gives $e^{\frac{\delta}{T}x} = \frac{K}{K_0}$

$$\therefore K = K_0 e^{\frac{\delta}{T}x} = \frac{P_1}{T} e^{\frac{\delta}{T}x} \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

For the upper section $K = K_1$ and $x = l \therefore K_1 = \frac{P_1}{T} e^{\frac{\delta}{T}l}$ (14)

We also have

$$W = \delta \int_0^l K dx = \delta \int_0^l \frac{P_1}{T} e^{\frac{\delta}{T}x}\,dx = P_1 \left(e^{\frac{\delta}{T}l} - 1\right) \ldots\ldots (15)$$

Example. What must be the upper section of a wrought-iron shaft of uniform resistance 1,000 ft. long, so that it will safely sustain its own weight and 75,000 lbs.?

Let $T = 10{,}000$ lbs., and
$\delta = 0.27$ lbs. per cubic inch.
Then Eq. (11) gives $K_0 = 7.5$ sq. inches, and
equation (14) gives $K_1 = 10.37$.

In these formulas *the form of section* does not appear. For tensile strains, the strength is practically independent of the form, but not so for compression. When it yields by crushing, the influence of form is quite perceptible, but not so much so as when it yields by bending under a compressive strain. The latter case will be considered under the head of flexure.

25. STRAINS IN A CLOSED CYLINDER.

If a closed cylinder is subjected to an *internal pressure*, it will tend to burst it by tearing it open along a rectilinear element, or by forcing the head off from the cylinder, by rupturing it around the cylinder. First, consider the latter case. The force which tends to force the head off is the total pressure upon the head, and the resisting section is the cylindrical annulus.

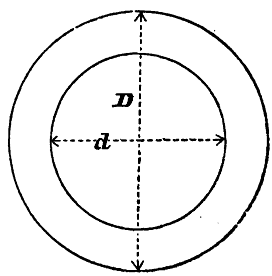


FIG. 11.

Let D = the external diameter,
d = the internal diameter,
p = the pressure per square inch, and
t = the thickness of the cylinder.

Then $\frac{1}{4}\pi d^2 p$ = the pressure upon the head,
$\frac{1}{4}\pi (D^2 - d^2)$ = the area of the cylindrical annulus,
$\frac{1}{4}\pi T(D^2 - d^2)$ = the resistance of the annulus, and
$2t = D - d$

Hence, for equilibrium,

$$\frac{1}{4}\pi d^2 p = \frac{1}{4}\pi T (D^2 - d^2)$$

$$\text{or, } d^2 p = 2Tt\,(D + d) = 4T(t^2 + dt) \ldots\ldots (16)$$

which solved gives $t = \left(-1 + \sqrt{1 + \frac{p}{T}}\right)\frac{d}{2} \ldots\ldots\ldots (17)$

Equation (16) may be written as follows:—

$$d^2 p = 4\,Tt\,(t + d),$$

and as t is generally small compared with d, we have $dp = 4Tt$ nearly.

Next consider the resistance to longitudinal rupturing. As it is equally liable to rupture along any rectilinear element, suppose that the cylinder is divided by any plane which passes through the axis. The normal pressure upon this plane is the force which tends to rupture it, and for a unit of length is

$$pd$$

and the resisting force is

$$2Tt,$$

hence, for equilibrium,

$$pd = 2Tt \ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

The value of t from (18) divided by that of t from (16) gives the ratio $\frac{D+d}{d}$, and since D always exceeds d, this ratio is greater than 2; hence there is more than twice the danger of bursting a boiler longitudinally than there is of bursting it around an annulus when the material is equally strong in both directions.

The last equation was established by supposing that all the cylindrical elements resisted equally, but in practice they do not; for, on account of the elasticity of the material, they will be compressed in the direction of the radius, thus enlarging the internal diameter more than the external, and causing a corresponding increase of the tangential stress on the inner over the outer elements. In a thick cylindrical annulus it is necessary to consider this modification.

To find the VARYING LAW OF TANGENTIAL STRAINS, let D and d be the external and internal diameters before pressure, and $D+z$ and $d+y$ the corresponding diameters after pressure. Then, as a first approximation—which is near enough for practice—suppose that the volume of the annulus is not changed, and we have

$$\tfrac{1}{4}\pi\,(D^2-d^2) = \tfrac{1}{4}\pi\,(D+z)^2 - \tfrac{1}{4}\pi\,(d+y)^2$$

$$\text{or, } Dz = dy \text{ nearly} \ldots\ldots\ldots\ldots\ldots\ldots \quad (19)$$

But the strain upon a cylindrical filament varies as its elongation divided by its length; see Eq. (3). Hence the strain on the external annulus, compared with the internal, is as

$$\frac{\pi(D+z)-\pi D}{\pi D} \text{ to } \frac{\pi(d+y)-\pi d}{\pi d}, \text{ or as } \frac{z}{D} \text{ to } \frac{y}{d}$$

which combined with (19) gives

$$\frac{d}{D^2} \text{ to } \frac{1}{d} \text{ or as } d^2 \text{ to } D^2, \text{ or as } r^2 \text{ to } R^2$$

where r and R are radii of the annulus.

Hence, *the strain varies inversely as the square of the distance from the axis of the cylinder.*

TO FIND THE TOTAL RESISTANCE, let

x = the variable distance from the axis of the cylinder,
T = the modulus of rupture, or of strain, and
t = the thickness of the annulus.

Then Tdx is the strain on an element at a distance r from the axis of the cylinder, or otherwise upon the inner surface of the cylinder; and according to the principle above stated, $T\frac{r^2}{x^2}dx$ is the strain on any element, and the total strain on both sides is

$$2Tr^2\int_r^R \frac{dx}{x^2} = 2T\frac{rt}{r+t} \quad \text{.................... (20)}$$

If $t = r$, this becomes

$$Tt$$

which compared with Eq. (18) shows that when the thickness equals the radius, the resistance is only half what it would be if the material were non-elastic. In (20) if t is small compared with r, it becomes $2\ Tt$ nearly, which is the same as equation (18).

If the ends of the cylinder are capped with hemispheres, the stress upon an elementary annulus at the inner surface is $2\pi\ Trdx$.* Proceeding as before, and we find that the total

* T. J. Rodman says *the resistance on any elementary annulus is $T\pi 2xdx$* ("Exp. on Metal for Cannon," p. 44); but it appears to me that, to make his expression correct, T must be the modulus at any element considered, and hence variable, whereas it should be constant. The strain on any elementry annulus whose

stress necessary to force the hemispherical heads off is

$$2\pi T \frac{r^2 t}{r+t} \dots\dots\dots\dots\dots\dots (21)$$

which is also the stress necessary to force asunder a sphere by internal pressure, when the elasticity is considered.

If cylinders are formed by riveting together plates of iron, their strength will be much impaired along the riveted section. The condition of the riveted joint will doubtless have much more to do with the strength than the compressibility of the material, and will hereafter be considered.

The preceding principles are especially applicable to homogeneous metals, where the thickness is considerable, such as cannons and spherical shells.

26. RESISTANCE OF GLASS GLOBES TO INTERNAL PRESSURE.

EXPERIMENTS OF WILLIAM FAIRBAIRN.

Description of the glass.	Diameter in inches.	Thickness in inches.	Bursting pressure in lbs. per square inch.	Bursting pressure in lbs. per square inch of section.
Flint-glass	4.0×3.98	0.024	84	3504
	4.0×3.98	0.025	93	3720
	4	0.038	150	3947
	4.5×4.55	0.056	280	5625
	6	0.059	152	3864
			Mean..........	4132
Green-glass..........	4.95×5.0	0.022	90	5113
	4.95×5.0	0.020	85	5312
	4.0 ×4.05	0.018	84	4666
	4.0 ×4.03	0.016	82	5126
			Mean..........	5054

distance is x from the centre of the sphere, is $T2\pi r dx, \frac{r^2}{x^2} = 2\pi r^3 T \frac{dx}{x^2}$; and the total resistance is the integral of this expression between the limits of r and $r+t$.

Crown-glass.........	4.2 × 4.35	0.025	120	5040
	4.05 × 4.2	0.021	126	6000
	5.9 × 5.8	0.016	69	6350
	6.0 × 6.3	0.020	86	6450

Mean...........5960

The following table exhibits the tensile strength of cylindrical glass bars according to the experiments of Mr. Fairbairn: —

Description of the glass.	Area of specimen in inches.	Breaking weight in lbs.	Tenacity per square inch.
Annealed flint-glass...	0.255	583	2286
	0.196	254	2540
Green-glass...........	0.220	639	2896
Crown-glass	0.229	583	2546

As might have been anticipated, the tenacity of bars is much less than globes; for it is difficult to make a longitudinal strain without causing a transverse strain, and the latter would have a very serious effect: it is also probable that the outer portion of the annealed glass is stronger than the inner, and there is a larger amount of surface compared with the section, in globes than in cylinders.

RIVETED PLATES.

27. RIVETED PLATES are used in the construction of boilers, roofs, bridges, ships, and other frames. It is desirable to know the best conditions for riveting, and the strength of riveted plates compared with the solid section of the same plates. The common way of riveting is to *punch* holes through both plates, into which red-hot bolts or rivets are placed, and headed down while hot. The process of punching strains, and hence weakens, the material. A better way is to *bore* the holes in the plates, and then rivet as before. The holes in the separate plates should be exactly opposite each other, so that there will be no side strain on the plates caused by driving the rivets home, and to secure the best effects of the rivets themselves. They are sometimes placed in single and sometimes in double rows, and experiment shows that the latter possesses great advantage over the former. Experiments have been made upon plates of

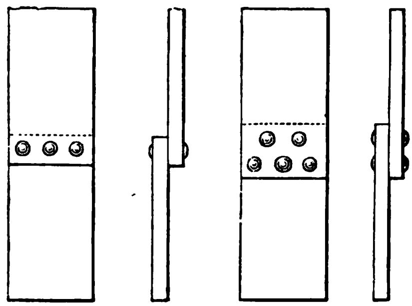


FIG. 12.

the form shown in Fig. 12, both with lap and butt-joints, and with single and double rows of rivets.*

Table showing the Strength of Single and Double Riveted Plates.

Cohesive strength of the plates in lbs. per square inch. T.	Strength of single-riveted joints of equal section to the plates, taken through the line of rivets. Breaking weight in lbs. per square inch.	Strength of double-riveted joints of equal section to the plates, taken through the line of rivets. Breaking weight in lbs. per square inch.
57,724	45,743	52,352
61,579	36,606	48,821
58,322	43,141	58,286
50,983	43,515	54,594
51,130	40,249	53,879
49,281	44,715	53.879
Mean..54,836	42,328	53,635

It will be observed that in double-riveting there is but little loss of strength, while there is considerable loss in single-riveting. In the preceding experiments the solid section of the plates, taken through the centre of the rivet-holes, was used; but, as Fairbairn justly remarks, we must deduct 30 per cent. for metal actually punched out to receive the rivets. But as only a few rivets came within the limits of the experiments, and as an extensive combination of rivets must resist more

* *London Phil. Transactions*, part 2d, 1850, p. 677.

effectually, and as something will be gained by the friction between the plates, it seems evident that we may use more than 60 per cent. of the strength of riveted plates as indicated above. Fairbairn says we may use the following proportions:—

Strength of plates	100
Strength of double-riveted plates	70
Strength of single-riveted plates	56

28. STRENGTH OF DRILLED AND PUNCHED BOILER PLATES.—A committee of the Railway Master Mechanics' Association for 1872 reported the following results of some experiments:—

Three pieces of $\frac{5}{16}$ inch boiler plate, $1\frac{3}{4}$ inch wide, were torn in two by hydraulic pressure.

No. 1 broke under a strain of............32,228 lbs.
No. 2 broke under a strain of............32,228 lbs.
No. 3 broke under a strain of............33,600 lbs.
The average breaking strain being....32,685 lbs.

Three pieces of $\frac{5}{16} \times 1\frac{3}{4}$ inch plate were *punched*, with a single $\frac{5}{8}$ inch hole in each piece. They were then subjected to a tensile strain, with the following result:—

No. 1 broke under a pressure of..........13,371 lbs.
No. 2 broke under a pressure of..........13,371 lbs.
No. 3 broke under a pressure of..........13,314 lbs.
The average being.................13,352 lbs.

Three pieces of $\frac{5}{16} \times 1\frac{3}{4}$ inch plate were *drilled*, with a single $\frac{5}{8}$ inch hole in each piece.

No. 1 broke under a pressure of..........17,828 lbs.
No. 2 broke under a pressure of..........17,485 lbs.
No. 3 broke under a pressure of..........17,622 lbs.
The average being.................17,645 lbs.

The average strength of the drilled plate being 4,163 lbs. greater than that of the punched plate.

Great care was taken to dress the pieces to the sizes given after they were punched or drilled.

The following comparative tests were then made with punched and drilled plates *riveted*.

Six pieces $1\frac{3}{4}$ inch wide, and cut from the same sheet as the foregoing, were punched and riveted together in pairs with the best $\frac{5}{8}$ inch rivets, one rivet to each pair, and were subjected to a tensile strain, with the following result:—

No. 1 broke in centre line of hole under...17,828 lbs.
No. 2 broke in centre line of hole under...17,828 lbs.
No. 3 broke in centre line of hole under...17,143 lbs.
The average breaking strain being....17,599 lbs.

Six pieces, duplicates of those last mentioned, were *drilled* and riveted together in pairs, one $\frac{5}{8}$ inch rivet to each pair.

No. 1 sheared the rivet under pressure of..17,143 lbs.
No. 2 sheared the rivet under pressure of..16,457 lbs.
No. 3 sheared the rivet under pressure of..15,428 lbs.
The average shearing strain being....16,342 lbs.

In the last set of experiments the strength of the plates was not determined, since the rivets broke by shearing before the plates broke. It is to be regretted that the size of the rivets was not increased sufficiently to cause the plates to break and thus secure a good comparative test. It is evident, however, that drilled holes cause the rivets to be sheared more easily than punched ones.

29. FAIRBAIRN'S RULE FOR THE SIZE AND DISTRIBUTION OF RIVETS.—The best size of the rivets, the distance between them, and the proper amount of lap of the plates, can be determined only by long experience, aided by experiments. Fairbairn gives the following table as the results of his information upon this important subject, to make the joint steam or water tight:—

Table showing the strongest Forms and best Proportions of Riveted Joints, as deduced from Experiments and actual Practice. ("*Useful Information for Engineers*," 1st Series, p. 285.)

Thickness of plates, in inches. t.	Diameter of the Rivets, in inches. d.	Length of rivets from the head, in inches. l.	Distance of rivets from centre to centre, in inches. a.	Quantity of lap in single-joints, in inches. b.	Quantity of lap in double-riveted joints, in inches. c.
$\frac{3}{16}$ to $\frac{4}{16}$	2 t	$4\frac{1}{2}$ t	6 t	6 t	10 t
$\frac{5}{16}$	"	"	5 t	"	"
$\frac{6}{16}$	"	"	"	$5\frac{1}{2}$ t	$8\frac{1}{4}$ t
$\frac{8}{16}$ to $\frac{12}{16}$	$1\frac{1}{2}$	"	4 t	$4\frac{1}{2}$ t	$6\frac{3}{4}$ t

30.—STRENGTH OF IRON IN DIFFERENT DIRECTIONS OF THE ROLLED SHEET.*

In obtaining specimens for these experiments, care was generally taken to have them cut in different directions of the rolling, longitudinally and transversely, and in some cases *diagonally*, to that direction. The table will be found to indicate the direction of slitting in each case, and the comparison contained in the table is given to show what information the inquiry has elicited.

The comparison is made principally on the *minimum* strength of each bar, being that which can alone be relied on in practice; for if the strength of the weakest point in a boiler be overcome, it is obviously unimportant to know that other parts had greater strength. In one case, however, two bars, one cut across the direction of rolling, and the other longitudinally, were, after being reduced to uniform size, broken up cold, with a view to this question. The result showed that the length-strip was $7\frac{7}{10}$ per cent. stronger than the one cut crosswise, considering the tenacity of the latter equal to 100. Of the other sets, embracing about 40 strips cut in each direction, it appears that some kinds of boiler iron manifest much greater inequality in the two directions than others. It is in certain cases not much over one per cent., and in others exceeds twenty, and as a mean of the whole series it may be stated to amount to six per cent. of the strength of the cross-cut bars. The number of trials on those cut diagonally is not perhaps sufficiently great to warrant a general deduction; but, so far as they go, they certainly indicate that the strength in this direction is less than either of the others.

Had we compared the mean instead of the least strength of bars as given in the table, the result would not have differed materially in regard to the relative strength in the respective directions.

The boiler-iron manufactured by Messrs. E. H. & P. Ellicott, which was tried in all these modes of preparation of specimens, gave the following results:—

1. When tried at *original sections*, seven experiments on length-sheet specimens gave a mean strength of 55285 lbs. per

* *Jour. of the Frank. Inst.*, Vol. 20, 2d series, p. 94. 1837.

square inch, the lowest being 44399 lbs., and the highest 59307 lbs. Fourteen experiments on cross-sheet specimens gave a mean of 53896 lbs., the lowest result being 50212 lbs., the highest 58839 lbs.; and six experiments on strips cut diagonally from the sheet exhibited a strength of 53850 lbs., of which the lowest was 51134 lbs., and the highest 58773 lbs.

2. When tried by filing notches on the edges of the strips, to remove the weakening effect of the shears, the *length-sheet* bars gave, at fourteen fractures, a mean strength of 63946 lbs., varying between 56346 lbs. and 78000 lbs. per square inch. The cross-sheet specimens tried after this mode of preparation exhibited, at three trials, a mean strength of 60236 lbs., varying from 55222 lbs. to 65143 lbs.; and the *diagonal* strips, at four trials, gave a mean result of 53925 lbs., the greatest difference being between 51428 lbs. and 56632 lbs. per square inch.

3. Of strips reduced to uniform size by filing, four comparable experiments on those cut lengthwise of the sheet gave a mean strength of 63947 lbs., of which the highest was 67378 lbs., and the lowest 60594 lbs.

From the foregoing statements it appears that by filing in notches and filing to uniformity, we obtain results 63946 lbs. and 63947 lbs. for the strength of strips cut lengthwise, differing from each other by only a single lb. to the square inch, and that by these two modes of preparation the cross-sheet specimens gave respectively 60236 lbs. and 60176 lbs., differing by only 60 lbs. to the square inch. This seems to prove that by both methods of preparing the specimens the accidental weakening effect of slitting had been removed by separating all that portion of the metal on which it had been exerted. Hence we may infer that the differences between length-sheet and cross-sheet specimens are really and truly ascribable to a difference of texture in the two directions, which will be seen to amount, in the case of filing in notches, to 6.15 per cent., and in that of filing to uniformity, to 6.26 per cent. of strength of cross-sheet specimens.

Table of the comparative view of the Strength of Specimens of ten different sorts of Boiler and one of Bar Iron, in the longitudinal, transverse, and diagonal direction of the rolling, as deduced from the least strength of each specimen, and the average minimum of each sort of Iron, in each direction in which it was tried.

No. of specimen referred to.	Strength in the longitudinal direction.	Strength in the transverse direction.	No. of specimen referred to.	Strength in the longitudinal direction.	Strength in the transverse direction.	Strength in the diagonal direction.
2	58977		125	57182	Tilted.	
3	53828		130	Tilted.	57789	
4	47167		133	do.	53176	
6		52280	135	do.		47738
8		50103	137	do.		50358
Mean.	53324	51191	Mean.	57182	55882	49048
42	51653	Puddled.	142	44399		
43	44102	do.	143	53135		
44	53836	do.	146	60594		
46	59262	H'd pla. *	148		52468	
48	59418	do.	149		52228	
49	57565	do.	150		56869	
51	H'd pla.	59656	151		53811	
53	H'd pla.	56062	152		56073	
56	Puddled.	57926	154			51134
58	do.	50570	157			52102
59	48308	Puddled.	160		53862	
60	58648	do.	162		50212	
61	52869	do.	164	56346		
62	57612	do.	167	56682		
64	Puddled.	45392	169	54361		
65	do.	51255	171			55612
68	57929	H'd pla.	174			51425
70	47638	do.				
71	H'd pla.	54634	Mean.	54253	53646	52568
73	do.	52657				
74	do.	49351				
Mean.	54074	53049	226		49053	
			227		53699	
			228		40643	
			229		46473	
			230	49368		
			Mean.	49368	47647	

The specimens from 42 to 74 were partly puddled iron, and partly Juniata blooms, hammered and rolled into plate. The

* Hammered and rolled into plates.

length and cross-sheet specimens of these two kinds must be compared separately.

All the experiments on No. 228 (cross) and 230 (length) were made at ordinary temperatures with a view to this comparison.

31.—TENSILE STRENGTH OF AMERICAN BOILER IRON, as determined by Mr. F. B. Stevens at the Camden and Amboy R. R. repair shops, New Jersey, by sixteen experiments upon high grade American boiler plate, gave the following results:—

Average breaking weight, lbs. per square inch	54,123
Highest breaking weight, lbs. per square inch	57,012
Lowest breaking weight, lbs. per square inch	51,813
Variation in *per centum* of highest	9.1

32.—TENSILE STRENGTH OF WROUGHT IRON AT VARIOUS TEMPERATURES.

Mr. Fairbairn has made experiments upon rolled plates of iron, and rods of rivet iron, at various temperatures. The former were broken in the direction of the fibre and across it. The specimen when subjected to experiment was surrounded with a vessel into which freezing mixtures were placed to produce the lower temperatures, and oil heated by a fire underneath to produce the high temperatures. The experiments were made upon Staffordshire plates, which are inferior to several other kinds in common use. The following table (A), gives a summary of these results.

The mean values given in the sixth column of this Table exhibit a remarkable degree of uniformity in strength for all temperatures, from 60 degrees to 395 degrees. The single example at 0 degrees gives a higher value than the mean of the others, but not higher than for some of the specimens at higher temperatures. At red heat the iron is very much weakened. This fact should be noticed in determining the strength of boiler-flues, as they are often subjected to intense heat when not covered with water.

The experiments upon rivet iron were made with the same machine, and in the same manner, the results of which are shown in the following table (B).

Table A—Showing the Resistance of Staffordshire Plates at different Temperatures.

No. of experiment.	Temp. Fahr.	Section of plate in inches.	Breaking weight in lbs.	Breaking weight per square inch, lbs.	Mean breaking weight per square inch, lbs.	Remarks.
1	0°	0.6868	33,660	49,009	49,009	With.
2	60	0.7825	31,980	40,357		Across.
3	60	0.6400	27,780	43,406	44,498	Across.
4	60	0.6368	31,980	50,219		With.
5	110	0.6633	29,460	44,160		Across.*
6	112	0.6800	28,620	42,088	42,291	With.
7	120	0.8128	37,020	40,625		With.
8	212	0.8008	31,980	39,935		With.
9	212	0.6633	30,300	45,680	45,005	Across.
10	212	0.6800	33,660	49,500		With.
11	270	0.6432	28,620	44,020	44,020	With.
12	340	0.6400	31,980	49,968	46,018	With.†
13	340	0.6800	28,620	42,088		Across.
14	395	0.6666	30,720	46,086	46,086	With.
15	Scarcely red	0.6200	23,520	38,032	34,272	Across.
16	Dull red	0.6076	18,540	30,512		Across.‡

Table B—Showing the Results of Experiments on Rivet Iron at different Temperatures.

No. of Experiment.	Temp. Fahr.	Section, inches.	Breaking weight in lbs.	Breaking weight per square inch, lbs.	Mean breaking weight per square inch, lbs.	Remarks.
17	—30°	0.2485	15,715	63,239	63,239	Too low.
18	+60	0.2485	15,400	61,971	62,816	Too low.
19	60	......	15,820	63,661		Too low.
20	114	......	17,605	70,845	70,845	
21	212	......	20,545	82,676		
22	212	0.1963	14,560	74,153	79,271	
23	212	0.2485	20,125	80,985		
24	250	0.1963	16,135	82,174	82,636	
25	270	0.2485	20,650	83,098		
26	310	0.1963	15,820	80,570	84,046	
27	325	0.1963	17,185	87,522		
28	415	0.2485	20,335	81,830	83,943	
29	435	......	21,385	86,056		
30	Red heat.	......	8,965	36,076	35,000	Too high.

* Too high; fracture very uneven.

† Too low; tore through the eye.

‡ Too high; the specimen broke with the first strain.

From this Table we see that there is a gradual increase of strength from 60 degrees to 325, where it appears to attain its maximum. The increase is a very important amount, being about 30 per cent.

It appears remarkable that the specimen at −30 degrees is stronger than the mean of the two at 60 degrees; but numerous experiments which have been made by different persons confirm this result, when the pieces are broken by a *steady strain.*

Experiments made by M. Baudrimont gave the following results:—*

Tenacity of Iron at 32° Fahr.	205	Kil. per square Millimetre,
Tenacity of Iron at 212° Fahr.	191	
Tenacity of Iron at 392° Fahr.	210	

in which we observe the same general results as in the preceding Tables.

But iron and steel will not resist shocks as effectually at very low temperatures as at moderate temperatures; as we shall have occasion to notice more particularly hereafter.

Mr. Johnson, when in the employ of the Navy Department, in 1844, made some experiments to determine the effects of thermo-tension upon different kinds of iron.† He took two bars of the same kind of iron, and of the same size, and broke one while cold. He then subjected the other to the same tension when heated 400 degrees, after which the strain was relieved, and the bar was allowed to cool, and the permanent elongation noted, after which it was broken by an additional load. It will thus be seen that the experiments were not conducted in the same way as those by Fairbairn. Table A, page 41, gives the results of his experiments.

Remarks.—From the two former sets of experiments, p. 39, it appears that the strength of the iron was increased by an increase of temperature at the time the bar was broken, and by the latter that it was not only increased, but, by being subjected to severe tension while at a high temperature, the increased strength was not lost by cooling. It hardly seems probable that this increased strength would be retained

* *Jour. Frank Inst.*, Vol. 20, 3d series, p. 344.

† *Senate Doc.*, No. 1, 28th Cong., 2d Sess., 1844–5, p. 639.

indefinitely, and hence it would be important to know how long it was after the piece was cooled before it was broken.

Table A—Results of Experiments on Thermo-Tension, at 400° Temperature.

KIND OF IRON.	Strength cold.	Strength after heated with thermo-tension.	Section.	Gain of length under thermo-tension with a strain equal to the strength when cold.	Gain of strength.	Total gain relative to the diminished section.
	Tons.	Tons.	Inches.	Per cent.	Per cent.	Per cent.
Tredegar, round...	60	71.4	1.91	6.51	19.00	25.51
Tredegar, round...	60	72.0	1.91	(6.51)	20.00	26.51
Tredegar, square bar	60	67.2	1.69	6.77	12.00	18.77
Tredegar, r'nd, No.3	58	68.4	1.15	5.263	17.93	23.19
Salisbury, round ..	105.87	121.0	3.59	3.73	14.64	18.37
Mean....................				5.75	16.64	22.40

These results are confirmed by the experiments of the committee of the Franklin Institute, as shown by the following Table.—See *Journal of the Franklin Institute*, Vol. 20, 3d Series, p. 22.

ABSTRACT OF TABLE

Of the comparative view of the Influence of High Temperatures on the strength of Iron, as exhibited by 73 experiments on 47 different specimens of that metal at 46 different temperatures, from 212° to 1317° Fahr., compared with the strength of each bar when tried at ordinary temperatures, the number of experiments at the latter being 163.

No. of the experiment.	Temperature observed at moment of fracture.	Strength at ordinary temperature.	Strength at the temperature observed.
1	212°	56736	67939
2	214	53176	61161
3	394	68356	71896
9	440	49782	59085
10	520	54934	58451
15	554	54372	61680
20	568	67211	76763
25	574	76071	65387
40	722	57133	54441
45	824	59219	55802
50	1037	58992	37764
58	1245	54758	20703
59	1317	54758	18913

Remark.—According to these experiments, as shown in the fourth column, the strength increases with the temperature to 394 degrees, when it attains its maximum; although in some cases the strength was increased by increasing the temperature to 568 degrees. By comparing the third and fourth columns we see that the strength is greater for all degrees from 212° to 574° than it is at ordinary temperatures, but above 574° it is weaker. The experiments on Salisbury iron showed that the maximum tenacity was 15.17 per cent. greater than their mean strength when tried cold. The committee above referred to determined the maximum strength of about half the specimens used in the preceding Table by actual experiment, and calculated it for the others; and from the results derived the following empirical formula for the diminution in strength below the maximum for high degrees of heat:—

$$D^5 = c\,(\theta - \dot{8}0)^{13}$$

in which D is the diminution after it has passed the maximum,
θ the temperature Fahrenheit, and
c a constant.

The value of the constant in empirical formulas is not strictly a *constant*, but is the mean of several values which are considered as constant. The value of the constant is found by substituting known values for all the other quantities in the equation.

This formula appears to be sufficiently exact for all temperatures between 520° and 1317°.

33.—TENSILE STRENGTH OF OTHER METALS AT DIFFERENT TEMPERATURES.—Experiments made by M. Baudrimont* showed that for all the following named metals the strength diminished as the temperature was increased; the results of which are given in Table A, page 43.

34.—EFFECT OF SEVERE STRAINS UPON THE ULTIMATE TENACITY OF IRON RODS.—Thomas Loyd, Esq., of England, took 20 pieces of 1⅝ S.C. bar iron, each 10 feet long, which were cut from the middle of as many rods. Each piece was cut into two parts of 5 feet each, and marked with the same letter. Those marked A, B, C, &c., were first broken, so as to get the average breaking strain. Those marked A2, B2, &c., were

* *Jour. Frank. Inst.*, Vol. 20, 3d Series, p. 344, 1850.

subjected to the constant action of three-fourths the breaking weight, previously found, for five minutes. The load was then removed, and the rods afterwards broken. The results are given in Table (B).

Table (A)—*Of the Mean Values of the Tenacity of the principal Malleable Metals at the temperature of 32°, 212° and 392° Fahrenheit.*

Name of the Metal.	Tenacity per square Millim. per cross section.		
	0°	212°	392°
Gold........................	18.400	15.224	12.878
Platina....................	22.625	19.284	17.277
Copper.....................	25.100	21.873	18.215
Silver......................	28.324	23.266	18.577
Palladium..................	36.481	32.484	27.077

Table B.—*Results of the Experiments.**

FIRST.		SECOND.	
Mark on the bars.	Breaking weight in tons (gross).	Mark.	Breaking weight in tons.
A	33.75	A 2	33.75
B	30.00	B 2	33.00
C	33.25	C 2	33.25
D	32.75	D 2	32.25
E	32.50	E 2	32.50
F	33.25	F 2	33.00
G	32.75	G 2	33.00
H	33.25	H 2	33.50
I	33.50	I 2	32.75
J	33.50	J 2	33.25
K	32 25	K 2	32.50
L	32.25	L 2	31.50
M	30.25	M 2	32.75
N	34.25	N 2	34.00
O	31.75	O 2	32.50
P	29.75	P 2	31.00
Q	33.50	Q 2	33.75
R	33.75	R 2	33.75
S	33.00	S 2	33.25
T	32.25	T 2	31.00
Mean..........	32.57		32.81

* Fairbairn, *Useful Information for Engineers*, First Series, p. 313.

We here see that a strain of 25 tons, or three-fourths the breaking weight, did not weaken the bar.

These experiments *indicate* that a frame may be subjected to a severe strain of three-fourths of its strength for a very short time without endangering its ultimate strength.

35. EFFECT OF REPEATED RUPTURE.—The following experiments were made at Woolwich Dockyard, England. The same bar was subjected to three or four successive ruptures by tensile strains. They show the remarkable fact that while great strains impair the elasticity, as shown by Hodgkinson, yet they do not appear to diminish the ultimate tenacity.

Table showing the effect of repeated Fracture on Iron Bars.

Mark.	First breakage.		Second breakage.		Third breakage.		Fourth breakage.		Reduced from sectional area of 1.37 sqr. inches to the following.
	Tons.	Stretch in 54 inches.	Tons.	Stretch in 36 inches.	Tons.	Stretch in 24 inches.	Tons.	Stretch in 15 inches.	
		In.		In.		In.			
A	33.75	0.9125	35.50	0.200					
B	33.75	0.9250	35.25	0.225	37.00	1.00	38.75		1.25
E	32.50	0.9250	34.75	0.125					
F	33.25	1.0500	35.50	0.112	37.25	0.62	40.40		1.18
G	32.75	0.8500	35.00	0.125	37.50		40.41		1.25
H	33.75	1.0625	36.25	0.187					
I	33.50	0.8375	34.50	0.62	36.50	1.50			
J	33.50	0.9250	36.00	0.025	36.75	1.12	41.75		1.25
L	32.25	Defect'e	36.50	0.150	37.75		41.00	0.31	1.25
M	30.25	Defect'e	36.50	.62	37.75	0.60	38.50	0.06	1.25
Mean	32.95		35.57		37.21		40.16		1.24
Mean per sq. in.	24.04		25.93		27.06		29.20		0.90

We thus see that while the section is reduced 10 *per cent., the strength is apparently increased over* 20 *per cent.* It is not, however, safe to infer that the strength is *actually* increased, for it is probable that it broke the first time at the weakest point, and the next time at the next weakest point, and so on.

We also observe that the total elongations are not proportional to the tensile strains, which is in accordance with the results of other experiments.

ANNEALED METAL—STRENGTH OF.

36. ANNEALING is a process of treating metals so as to make them more ductile. To secure this, the metals are subjected to a high heat and then allowed to cool slowly. *Steel* is softened in this way, so that it may be more easily worked. Campin* says that *steel* should not be overheated for this purpose. Some bury the heated steel in lime; some in cast-iron borings; and some in saw-dust. He (Campin) says the best plan is to put the steel into an iron box made for the purpose, and fill it with dust-charcoal, and plug the ends up to keep the air from the steel; then put the box and its contents into a fire until it is heated thoroughly through, and the steel to a low red heat. It is then removed from the fire, and the steel left in the box until it is cold. Tools made of annealed steel will, in some cases, last much longer than those made of unannealed steel.

But it appears from the following Table that it weakens *iron* to anneal it.

Table of the Strength of Wrought Iron Annealed at Different Temperatures.†

No. of comparisons.	Strength at ordinary temperature before annealing.	Temperature at which annealing took place.	Strength at the annealing temperature.	Strength after annealing and cooling.	Ratio of diminution of strength.
1	57,133	1037°	37,764	55,678	0.025
5	53,774	1155	21,967	45,597	0.152
10	52,040	1245	20,703	38,843	.253
15	48,407	Bright welding heat.	......	38,676	.201
17	73,880	Low welding heat.	......	54,578	.275
18	76,986	Bright welding heat.	......	50,074	.349
19	89,162	Low welding heat.	......	48,144	.460

37. THE STRENGTH OF IRON AND STEEL ALSO DEPENDS largely upon the process of their manufacture and their treatment afterwards. The strength of wrought iron depends upon the ore of which it is made; the manner in which it is smelted and puddled; the temperature at which it is hammered, and the amount of hammering which it receives in bringing it into

* Campin's *Mechanical Engineering.*

† *Jour. Frank. Inst.*, Vol. 20, 2d Series, p. 109, 1837.

shape. The same remark applies to cast-steel. If the former is hammered when it is comparatively cold, it will weaken it, especially if the blows are heavy; but the latter, steel, may be greatly damaged, or even rendered worthless by excessive heat, and it is greatly improved by hammering when comparatively cold. For the effect of tempering on the crushing strength, see Article 59.

Different ores with essentially the same treatment produce essentially different iron. Thus, the Lake Superior ores, near Marquette, make a soft but very tough iron. Some of the strongest specimens of iron which have been made in this country were made from these ores, but it is found that the elastic limit is passed with a much less strain compared with its ultimate strength than many other irons. Manufacturers, therefore, mix it with other ores so as to raise the elastic limit. They often mix it with cheaper ores so as to cheapen the product. They also mix it with cheaper ores so as to improve the quality of the iron which would result if cheap ores only were used. The mixing of ores from various mines is constantly going on among manufacturers for various reasons. In this way is secured irons of various grades of hardness, of elasticity, of *weldibility*, and of tenacity.

There is even a greater difference in the quality of steel than of wrought-iron. We have the well-known classes of blister steel, crucible steel, Bessemer steel, and more recently of chrome steel. Uniformity of product is more earnestly sought in the manufacture of the several grades of steel than of iron, but when the same iron is used by the same person, under the same conditions, so far as he is able to control them, the expert finds that there is a perceptible difference in the products. Some steel takes a higher temper than others; some is softer; some more brittle; some more tenacious; some is better to resist crushing; some better for sharp tools; some better to work into masses than others; and it is often necessary for those who use steel to become acquainted with the *grade* which will best suit their purpose.

Although experts may detect differences in steels of the same general *grade*, yet manufacturers are able to produce steel having given general characteristics so uniform that the common workman will not detect any difference in them.

38. CHROME STEEL. This is a peculiar product, which, according to the older definitions of steel (depending upon a certain per cent. of carbon), is not steel, but which possesses many of the characteristics of steel. The manufacturers of it claim that they can produce a steel of more uniform quality of any particular grade, especially in large masses, than can be produced by carbon steel.

The tensile strength exceeds considerably that of the best crucible steel (excepting the remarkable specimen noted on p. 25). The experiments which were made upon twelve specimens of tool steel, which were cut from three bars, at the West Point Foundry, gave the following results:—

Highest strength........198,910 lbs. per square inch.
Lowest strength........163,760 lbs. per square inch.
Average of all..........179,980 lbs. per square inch.

The limit of elastic resistance is also high, being more than half of its ultimate strength.*

39. PROLONGED FUSION OF CAST IRON.—Cast iron is also subjected to great modifications of strength on account of the manipulations to which it is or may be subjected in its manufacture and preparations for use. The strength in some cases is greatly increased by keeping the metal in a fused state some time before it is cast. Major Wade made experiments upon several kinds of iron, all of which were increased in strength with prolonged fusion (see *Rep.*, p. 44), one example of which is given in the following

Table showing the Effects of Prolonged Fusion.

	Tensile Strength in lbs. per sq. in.
Iron in fusion..................½ hour	17,843
Iron in fusion..................1 hour	20,127
Iron in fusion..................1½ hour	24,387
Iron in fusion..................2 hours	34,496

40. EFFECT OF REMELTING CAST IRON.—But the greatest effect was produced by remelting. The density, tenacity,

* Report of Capt. J. B. Eads, C.E. This steel is used in the construction of the noted St. Louis bridge.

and transverse strength were all increased by it, within certain limits. For instance, a specimen of No. 1 Greenwood pig-iron gave the following results. (*Rep.*, p. 279.)

Table showing the Effects of Remelting.

No. 1 Greenwood Iron.	Specific gravity.	Tensile Strength.
Crude pig-iron	7,032	14,000
Crude remelted once	7,086	22,900
Crude remelted twice	7,198	30,229
Crude remelted three times	7,301	35,786

But there is a point beyond which remeltings will weaken the iron. Mr. Fairbairn made an experiment in which the strength of the iron was increased for twelve remeltings, and then the strength decreased to the eighteenth, where the experiment terminated. In some cases no improvement is made by remelting, but the iron is really weakened by the process; so that it becomes necessary to determine the character of each iron under the various conditions by actual experiment.

The laws which govern Greenwood iron were so thoroughly determined that the results which will follow from any given course of treatment may be predicted with much certainty (*Rep.*, p. 245).

By mixing grades Nos. 1, 2, and 3, and subjecting them to a third fusion, *one* specimen was obtained whose density was 7,304, and whose tenacity was 45,970 lbs., which is the strongest specimen of cast iron ever tested. (*Rep.*, p. 279.)

As a general result of these experiments, Major Wade remarks (p. 243), "that the softest kinds of iron will endure a greater number of meltings with advantage than the higher grades. It appears that when iron is in its best condition for casting into proof bars (that is, small bars for testing the metal) of small bulk, it is then in a state which requires an additional fusion to bring it up to its best condition for casting into the massive bulk of cannon."

41. THE MANNER OF COOLING also affects the strength. It was found that the tensile strength of large masses was increased by slow cooling; while that of small pieces was increased by rapid cooling. (*Rep.*, p. 45.)

42. THE MODULUS OF STRENGTH IS MODIFIED, we thus see, by a great variety of circumstances; and hence it is impossible to assign any *arbitrary* value to it for any material that will be both safe and economical; but its value must be determined, in any particular case, by direct experiment, or something in regard to the quality of the material must be known before its approximate value can be assumed.

43. SAFE LIMIT OF LOADING.—Structures should not be strained so severely as to damage their elasticity. According to Article 9, it appears that a weight suddenly applied will produce twice the elongation that it will if applied gradually or by increments. Hence, structures which are subjected to shocks by sudden applications of the load, should be so proportioned as to resist more than double the load as a constant dead-weight without straining it beyond the elastic limit.

This method of indicating the limits is perfectly rational; but, unfortunately, the elastic limits have not been as closely observed and as thoroughly determined by experimenters as the limit of rupture. The latter was formerly considered more important, and hence furnished the basis for determining the safe limit of the load. Observations on good constructions have led engineers to adopt the following values as mean results for permanent strains in bars:—

For wood,	$\frac{1}{10}$	of the load which would produce rupture.
For wrought iron,	$\frac{1}{3}$ to $\frac{1}{5}$	
For cast iron,	$\frac{1}{4}$ to $\frac{1}{6}$	

Further observations will be made upon this subject in the latter part of the volume.

CHAPTER II.

COMPRESSION.

ELASTIC RESISTANCE.

44.—COMPRESSION OF CAST IRON.—Captain T. J. Rodman, in his Report upon metals for cannon, page 163, has given the results of experiments upon a piece of cast iron, which was taken from the body of the same gun as was the specimen referred to on page 11 of this work, the results of which are given in the following Table.

TABLE

Showing the Compression, permanent Set; and coefficient of Elasticity of a solid Cylinder* 10 *inches long and* 1.382 *inch diameter.*

Weight per square inch of section in lbs.	Compression per inch of length.	Permanent set per inch of length.	Coefficient of elasticity.
1,000	0.000090	0.	11,111,000
2,000	0.000170	0.	11,824,000
3,000	0.000255	0.000005	11,843,100
4,000	0.000320	0.000015	12,500,000
5,000	0.000385	0.000025	12,987,000
6,000	0.000455	0.000030	13,189,000
7,000	0.000505	0.000035	13,861,300
8,000	0.000575	0.000045	13,813,000
9,000	0.000645	0.000055	13,952,000
10,000	0.000705	0.000070	14,196,000
15,000	0.001035	0.000170	14,492,000
20,000	0.001395	0.000300	14,337,000
25,000	0.001825	0.000495	13,687,900
30,000	0.002380	0.000820	12,602,300

We observe that the coefficient of elasticity is much less for the first strains than for those that follow. It thus appears that this metal resists more strenuously after it has been somewhat compressed than at first. The coefficient of elasticity is con-

* The author computed the coefficients of elasticity from the other data of the table.

siderably less than for the corresponding piece, as given on page 11. The difference is very much greater than that found by Mr. Hodgkinson in the specimens which he used in his experiments. He took bars 10 feet long, and about an inch square, and fitted them nicely in a groove so that they could not bend, and occasionally, during the experiment, they were slightly tapped to avoid adherence. The metal was the same kind as that used in the experiment recorded on page 13.

TABLE

Giving the Results of Experiments by Mr. Hodgkinson on bars of Cast Iron ten feet long.

Pressure per square inch of section. P.	Compression per inch of length.		Coefficient of elasticity per square inch.	Error in parts of P of the formula $P=170,763\lambda_c - 36,318\lambda_c^2$
	Total. λ_c	Permanent.		
lbs.	in.	in.	lbs.	
2064.74	0.0001561	0.00000391	13,231,300	$-\frac{1}{36}$
4129.49	0.0003240	0.00001882	12,764,910	$-\frac{1}{816}$
6194.24	0.0004981	0.00003331	12,442,300	$+\frac{1}{52}$
8258.98	0.0006565	0.00005371	12,585,100	$+\frac{1}{1330}$
10323.73	0.00082866	0.00007053	12,467,100	$+\frac{1}{312}$
12388.48	0.00100250	0.00009053	12,357,200	$+\frac{1}{248}$
14453.22	0.00128025	0.00011700	12,253,700	$+\frac{1}{179}$
16517.97	0.00136150	0.00014258	12,141,200	$+\frac{1}{139}$
18582.71	0.00154218	0.00017085	12,058,100	$+\frac{1}{161}$
20647.46	0.00171866	0.00020685	12,021,800	$+\frac{1}{629}$
24776.95	0.00208016	0.00036810	11,920,000	$-\frac{1}{176}$
28906.45	0.00247491	0.00045815	11,687,400	$-\frac{1}{275}$
33030.80	0.0029450	0.00050768	11,222,750	$+\frac{1}{64}$
37159.65	0.003429			

In this case the highest coefficient of elasticity results from the smallest strain which is recorded. The difference in this respect between this example and the preceding one results doubtless from the internal structure of the iron. The coefficient in both these cases is much less than that found for other kinds of cast iron, as is shown in the Table of Resistances in the Appendix.

Mr. Hodgkinson proposed the empirical formula, $P = 170,763\lambda_c - 36,318\lambda_c^2$, to represent the results of the experiments; and although it may represent more nearly the results for a greater range of strains than equation (3), yet there is no advantage in its use in practice.

45.—COMPRESSION OF WROUGHT IRON.

Mr. Hodgkinson also made experiments upon bars of wrought iron in precisely the same manner as upon those of cast iron, the results of which are given in the following

TABLE,

*Giving the Results of Experiments by Mr. E. Hodgkinson on bars of Wrought Iron, each of which was ten feet long.**

Weight producing the compression.	1st Bar. Section = 1.025 × 1.025 sq. in.		2d Bar. Section = 1.016×1.02 sq. in.	
	Amount of Compression.	Value of E.	Amount of Compression.	Value of E.
lbs.	inch.	lbs.	inch.	lbs.
5098	0.028	20,796,500	0.027	21,864,000
9578	0.052	21,049,000	0.047	23,595,000
14058	0.073	21,979,000	0.067	24,273,000
16298	0.085	21,343,000		
18538	0.096	22,156,000	0.089	24,108,000
20778	0.107	22,160.000	0.100	24,038,000
23018	0.119	23,587,000	0.113	23,587.000
25258	0.130	22,095,000	0.128	23,679,000
27498	0.142	22,111,000	0.143	22,259,000
29738	0.152	21,938,000	0.163	21,139,000
31978	0.174	20,979,000	0.190	19,478,000
In ½ hour.			0.261	
Again after ½ hour.			0.269	
Then repeated.			0.328	

46. GRAPHICAL REPRESENTATION.—These two cases are graphically represented in Fig. 13. It is seen from the tables that the compressions are quite uniform for a large range of strains, and hence equation (2), page 17, is applicable to compressive strains when within the elastic limits. In the case of the wrought-iron bars, the first one attains its maximum coefficient of elasticity for a strain somewhat less than one-half its ultimate resistance to crushing, and the second bar at about one-third its ultimate resistance.

47. COMPARATIVE RESISTANCE OF CAST AND WROUGHT IRON.—The coefficient of elasticity is a measure of the compressibility of metals. Hence, an examination of the two preceding Tables shows that of the specimens used in these

* The coefficients of elasticity were computed by the author.

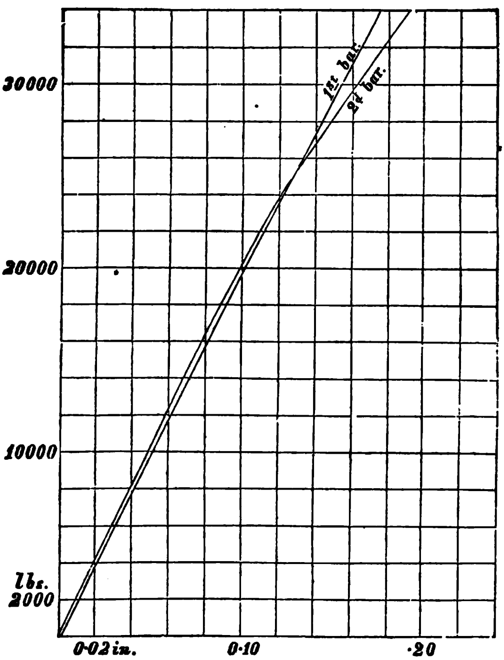


FIG. 13.

experiments, the cast iron was compressed nearly twice as much as the wrought iron for the same strains. An examination of the Table of Resistances, in the Appendix, shows that for a mean value wrought iron is compressed about two-thirds as much as cast iron for the same strain. The same ratio evidently holds for tension. This is contrary to the popular notion that cast iron is *stiffer* than wrought iron; for it follows from the above that a cast-iron bar may be stretched more, compressed more, and bent more, than an equal wrought iron one with the same force under the same circumstance, and in some cases the changes will be twice as great. One reason why cast is considered stiffer than wrought iron probably is, that wrought iron does not fail suddenly as a general thing, but it can be seen to bend for a long time after it begins to break; while cast iron, on account of its granular structure, fails suddenly after it begins, and the bending which has previously taken place is not noticed. It is not safe to trust to such general observations for scientific or even practical purposes, but careful observations must be made, so that all the circumstances of the case may be definitely known. It will hereafter be shown that the ultimate resistance to crushing of cast iron is double that of wrought iron, and yet Fairbairn and other English engineers have justly insisted upon the use of wrought iron for tubular and other bridges. For, without considering the comparatively treacherous character of cast iron when heavily loaded, it appears that within the elastic limits (and the structure should not be loaded to exceed that), a wrought iron structure is *stiffer* than a cast iron one of the same dimensions, and will sustain more within the elastic limits for a given compression, extension, or deflection.

48. COMPRESSION OF STEEL.—Good cast steel has a higher coefficient of elasticity than any other metal upon which experiments have been made for the purpose of determining it; and yet it exceeds by only a small amount the coefficient for the best iron. But the *limit of elasticity* of steel *greatly* exceeds that of iron, as has already been observed in article 38. In the noted St. Louis Bridge the coefficient of elasticity of the steel was not to be less than 26,000,000 lbs., nor exceed 30,000,000 lbs.

49. COMPRESSION OF OTHER MATERIALS.—All materials are compressible as well as extensible, and it is generally assumed that their resistance to compression, within the elastic limits, is the same as for extension; but, as has been seen in the previous articles, this is not rigorously correct. The mean value, however, of the coefficient of elasticity is sufficiently exact for practical cases.

50.—EXAMPLE.—1. Required the compression of a sphere which rests upon a plane; the weight of the sphere being the only load.

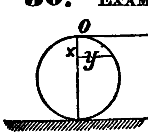


FIG. 14.

This may be readily solved by supposing that all horizontal sections before compression remain plane and horizontal during compression.

Take o, the highest point of the sphere, as the origin of co-ordinates; x vertical and y horizontal.

$\delta =$ the weight of a unit of volume; and

$r =$ the radius of the sphere.

Then $y^2 = 2\,r\,x - x^2$;

$K = \pi y^2 = \pi (2rx - x^2) =$ the area of any horizontal section; and

$$P = \delta \int K\,dx = \delta\pi \int_0^x (2rx - x^2)\,dx = \delta\pi r x^2 - \tfrac{1}{3}\delta\pi x^3 =$$

the weight of all the segment above the section.

$$\therefore \lambda = \int_0^x \frac{P\,dx}{EK} = \frac{\delta}{E}\int_0^x \frac{\pi r x^2 - \frac{1}{3}\pi x^3}{\pi(2rx - x^2)}\,dx =$$

$$\frac{\delta}{E}\int_0^x \frac{r\,x\,dx}{2r - x} - \frac{1}{3}\,\frac{\delta}{E}\int_0^x \frac{x^2\,dx}{2r - x} =$$

$$\frac{1}{6}\frac{\delta}{E}\left(x^2 - 2rx + 4r^2\,Nap.\,log.\frac{2r}{2r - x}\right)$$

For a hemisphere this becomes, by making $x = r$;

$$\lambda = \frac{r^2\delta}{6E}(4\,Nap.\,log.\,2 - 1) = 0.29543\,\frac{\delta r^2}{E}$$

For the sphere $x = 2r$ and the Equation gives

$$\lambda = \infty$$

as it should, since *theoretically* there is only a point where it touches the plane to sustain the whole sphere, and Eq. (2) gives $\lambda = \infty$ when $k = 0$, and the other quantities are finite. But *practically* we know that this is not true, and it is easily accounted for by supposing that the sphere as well as the body upon which it rests is flattened in the vicinity of x so as to present a surface of finite magnitude for supporting the weight above it.

51. Required the compression of any portion of a cylindrical annulus when it lies upon a horizontal plane, its axis being parallel to the plane, and the weight of the annulus being the only load.

The true distortion in this case, as well as in the preceding one, is peculiar. There will be a bulging outward, as well as depression vertically, and there will also be a *moment* of stress. But it may be solved by assuming that the only strains are vertical compressions, and that horizontal sections remain horizontal during compression.

If the annulus is very thin compared with the diameter of the circle we have, the origin being at the highest point, x vertical and y horizontal;

$t =$ the thickness;

$P = t\,\delta\,\text{vers sin}^{-1}\frac{x}{r}$;

$l = dx =$ the height of an infinitely short prism; and

$K = t\,\frac{r}{y} =$ the horizontal section at any point for a unit of length.

$$\therefore \lambda = \frac{\delta}{r\,E} \int_0^x \sqrt{2\,r\,x - x^2}\ \text{vers sin}^{-1} \frac{x}{r}\,dx$$

The approximate value of which may be found by developing it into a series and integrating several terms. If the origin of co-ordinates be taken at the centre of the cylinder we have

$$\lambda = \frac{\delta}{rt\,E} \int_0^x \sqrt{r^2 - x^2}\,\cos^{-1} \frac{x}{r}\,dx$$

From this example we see that if a large cylinder (as for instance a steam cylinder, or a boiler) be made exactly cylindrical when it stands upon one end, it will be oval when it is placed on its side.

52. THE PARALLELISM OF SECTIONS, which was assumed in the two preceding problems, would not be realized in any actual case. The solution properly belongs to the *Mathematical Theory of Elasticity*, and involves the most refined analysis. An *exact* solution may not be possible.

53. A GENERAL STATEMENT OF THE PROBLEM of the mathematical theory of the equilibrium of a solid body is:

A solid of any shape, when undisturbed, is acted on in its substance by a force distributed through it in any manner, and displacements are arbitrarily produced. It is required to find the displacement of every point of its surface.

54. ANALYTICAL EXPRESSION. If X, Y, and Z be the resolved components of the applied force, and the remaining notation be as given on page 217, then for any point (x, y, z) within the solid we have

$$\frac{dp_{xx}}{dx} + \frac{dp_{xy}}{dy} + \frac{dp_{xz}}{dz} + X = 0$$

$$\frac{dp_{yx}}{dx} + \frac{dp_{yy}}{dy} + \frac{dp_{yz}}{dz} + Y = 0$$

$$\frac{dp_{zx}}{dx} + \frac{dp_{zy}}{dy} + \frac{dp_{zz}}{dz} + Z = 0$$

55. PARTICULAR VALUES. In the preceding problems $X = 0$, $Y = 0$, and $Z = -g$. It will be shown in Chapter IX. that when a prismatic bar is compressed by a longitudinal stress that it will expand laterally, and in a perfectly homogeneous body the expansion per unit will be approximately $\frac{1}{4}$ of the contraction per unit.

ULTIMATE STRENGTH.

56. MODULUS FOR CRUSHING.—The modulus of resistance to crushing is the pressure which is necessary to crush a piece of any material whose section is unity, and whose length does not exceed from one to five times its diameter.

The *law of resistance* to crushing is not simple. Granular blocks, like some kinds of stone and cast iron, often separate in planes (or surfaces approximating to planes) which are inclined to the base.

Glass in some cases separates in thread-like filaments when it is crushed. Wrought iron does not fail suddenly, like the bodies just mentioned, but considerable tenacity remains between the fibres after it begins to fail. Then, too, in all cases the resistance to crushing depends upon the length of the piece. If the blocks are very short (from one to five times the diameter as mentioned above) we get *simply* crushing; but if they are long compared with their diameter, the phenomena are very complex, there being a combination of bending and crushing, and the law which governs it is determined only approximately by direct experiment, as indicated in article 62.

It is found by experiment that the resistance of short pieces (blocks) to crushing varies nearly as the transverse section of the piece, no matter what the form of the fracture may be.

Hence, if

$P =$ the crushing force, and
$K =$ the section under pressure, we have

$$P = CK. \qquad (22)$$

MODULUS OF STRAIN.—If the force P is not sufficient to crush the piece, we have for the *strain on a unit of section*

$$C_1 = \frac{P}{K}. \qquad (23)$$

57.—RESISTANCE TO CRUSHING OF CAST-IRON.

TABLE

*Of the results of Experiments on the Tensile and Crushing Resistances of Cast-Iron of various kinds, made by Eaton Hodgkinson.**

Description of the Iron.	Tensile Strength per square inch. T.	Height of Specimen.	Crushing strength per square inch. C.	Ratio of Tenacity to crushing. $C \div T$.	
	Lbs.	inch.	Lbs.		Mean.
Low Moor Iron, No. 1.	12,694	¾ 1½	64,534 56,445	1 : 5.084 1 : 4.446	1 : 4.765
" " No. 2.	15,458	¾ 1½	99,525 92,332	1 : 6.438 1 : 5.973	1 : 6.205
Clyde Iron, No. 1.....	16,125	¾ 1½	92,869 88,741	1 : 5.759 1 : 5.503	1 : 5.631
" " No. 2.....	17,807	¾ 1½	109,992 102,030	1 : 6.177 1 : 5.729	1 : 5.953
" " No. 3.....	23,468	¾ 1½	107,197 104,881	1 : 4.568 1 : 4.469	1 : 4.518
Blaenavon Iron, No. 1.	13,938	¾ 1½	90,860 80,561	1 : 6.519 1 : 5.780	1 : 6.149
" " No. 2.	16,724	¾ 1½	117,605 102,408	1 : 7.032 1 : 6.123	1 : 6.577
" " No. 3.	14,291	¾ 1½	68,559 68,532	1 : 4.797 1 : 4.795	1 : 4.796
Calder Iron, No. 1.....	13,735	¾ 1½	72,193 75,983	1 : 5.256 1 : 5.532	1 : 5.394
Coltness Iron, No. 3...	15,278	¾ 1½	100,180 101,831	1 : 6.557 1 : 6.665	1 : 6.611
Brymbo Iron, No. 1...	14,426	¾ 1½	74,815 75,678	1 : 5.186 1 : 5.264	1 : 5.216
" " No. 3...	15,508	¾ 1½	76,133 76,958	1 : 4.909 1 : 4.963	1 : 4.936
Bowling, No. 2.......	13,511	¾ 1½	76,132 73,984	1 : 5.635 1 : 5.476	1 : 5.555
Ystalyfea, No. 2...... (Anthracite)	14,511	¾ 1½	99,926 95,559	1 : 6.886 1 : 6.585	1 : 6.735
Yniscedwyn, No. 1.... (Anthracite)	13,952	¾ 1½	83,509 78,659	1 : 5.985 1 : 5.638	1 : 5.811
" No. 2.....	13,348	¾ 1½	77,124 75,369	1 : 5.778 1 : 5.646	1 : 5.712
Stirling, 2d quality....	25,764	¾ 1½	125,333 119,457	1 : 4.865 1 : 4.637	1 : 4.751
" 3d quality....	23,461	¾ 1½	158,653 129,876	1 : 6.762 1 : 5.536	1 : 6.149
Mean........	16,303	.. ..	88,800 94,730	Mean ratio 1 : 5.64	

* *Supplement to Bridges*, by Geo. R. Brunell, and Wm. T. Clark. John Weale, London.

In this table the ratio of resistances range from about $4\frac{1}{2}$ (Clyde, No. 3) to more than 7 (Blaenavon, No. 2). The same experimenter once obtained the ratio of 8.493 from a specimen of Carron iron, No. 2, hot blast; * and the mean of several experiments, made at the same time, gave 6.594. Hence we have, as the mean result of a large number of experiments, that the crushing resistance of cast iron is about 6 times as great as its tenacity; but the extremes are from $4\frac{1}{2}$ to $8\frac{1}{2}$ times its tenacity.

58. RESISTANCE OF WROUGHT IRON TO CRUSHING.—Comparatively few experiments have been made to determine how much wrought iron will sustain at the point of crushing, and those that have been made give as great a range of results as those for cast iron.

Hodgkinson gives $C = 65000$ †
Rondulet " $C = 70800$ ‡
Weisbach " $C = 72000$ §
Rankine " $C = 30000$ to 40000 ‖

It is generally assumed that wrought iron will resist about two-thirds as much to crushing as to tension, but the experiments fail to give a very definite ratio.

59. RESISTANCE OF STEEL TO CRUSHING.—Major Wade found the following results from experiments upon the several samples of cast steel, all of which were cut from the same bar and treated as indicated in the table.¶

Specimen.	Length.	Diameter.	Crushing in lbs. per sq. inch.
Not Hardened	1.021	0.400	198,944
Hardened, low temper	0.995	0.402	354,544
" mean "	1.016	0.403	391,985
" high " for tools for turning hard steel.	1.005	0.405	372,598

* *Résistance des Matériaux*, Morin, p. 95.
† *Vose, Handbook of Railroad Construction*, p. 127. (Old Edition.)
‡ *Mahan's Civil Engineering*, p. 97.
§ *Weisbach, Mech. and Eng.*, vol. i., p. 215.
‖ *Rankine's Applied Mech.*, p. 633.
¶ Report on *Metals for Cannon*, p. 258.

CHROME STEEL resists from 160,000 to 195,000 pounds per square inch. Indeed Captain Eads says in his report that the crushing resistance of chrome steel may be increased to any desirable amount by the simple addition of chromium.

60. RESISTANCE OF WOOD TO CRUSHING.—The resistance of wood to crushing depends as much upon its state of dryness, and conditions of growth and seasoning, as its tenacity does. The following are a few examples:—

Kind of Wood.	Moderately Dry.	Very Dry.
Ash	8,680	9,360
Oak (*English*)	6,480	10,058
Pine (*Pitch*)	6,790	6,790

These results, compared with the corresponding numbers in article 22, show that these kinds of wood will resist from $1\frac{1}{2}$ to nearly 2 times as much to tension as to compression. For other examples see the Table in the appendix.

61. RESISTANCE OF GLASS TO CRUSHING.—We owe most of our knowledge of the strength of glass to Wm. Fairbairn and T. Tate, Esq. According to their experiments we have the following results for the crushing resistance of specimens of glass whose heights varied from one to three times their diameter.

MEAN CRUSHING RESISTANCE OF CUT-GLASS CUBES AND ANNEALED GLASS CYLINDERS.

Description of the Glass.	Weight per Square Inch.	
	Cubes.	Cylinders.
	lbs.	lbs.
Flint Glass	13,130	27,582
Green Glass	20,206	31,876
Crown Glass	21,867	31,003
Mean	18,401	30,153

The ratio of the mean of the resistances is as 1 to 1.6 nearly. The cylinders were cut from round rods of glass, and hence retained the outer skin, which is harder than the interior, while the cubes were cut from the interior of large specimens. This may partially account for the great difference in the two sets of experiments. The cubes gave way more gradually than the cylinders, but both fractured some time before they entirely failed. The cylinders failed very suddenly at last, and were divided into very small fragments. The specimens had rubber bearings at their ends, so as to produce an uniform pressure over the whole section.

62. STRENGTH OF PILLARS.—The strength of pillars for *incipient flexure* has been made the subject of analysis by Euler and others, but practical men do not like to rely upon their results. Mr. Hodgkinson deduced empirical formulas from experiments which were made upon pillars of wood, wrought iron, and cast iron. The experiments were made at the expense of Wm. Fairbairn, and the first report of them was made to the Royal Society, by Mr. Hodgkinson, in 1840. The following are some of his conclusions:—

1st. In all long pillars of the same dimensions, when the force is applied in the direction of the axis, the strength of one which has flat ends is about three times as great as one with rounded ends.

2d. The strength of a pillar with one end rounded and the other flat, is an arithmetical mean between the two given in the preceding case of the same dimensions.

3d. The strength of a pillar having both ends firmly fixed, is the same as one of half the length with both ends rounded.

4th. The strength of a pillar is not increased more than $\frac{1}{7}$th by enlarging it at the middle.

To determine general formulas, bars of the same length and different sections were first used; then others, having constant sections and different lengths; and formulas were deduced from the results. The formulas thus made were compared with the results of experiments on bars whose dimensions differed from the preceding. The following are the results of some of his

EXPERIMENTS ON SQUARE PILLARS.

Length of the bars.	Side of the square.	Crushing weight.	Exponent of the side.
Feet.	Inches.	Lbs.	
10	0.766 1.51	1.948 23,025	3.57
10	1.00 1.50	4,225 23,025	4.17
$7\frac{1}{2}$	1.02 1.53	10,236 45,873	3.69
$7\frac{1}{2}$	0.50 1.00	583 9,873	4.08
5	0.50 1.00	1,411 18,038	3.67
$2\frac{1}{2}$	0.502 1.00	4,216 27,212	2.69
$2\frac{1}{2}$	0.502 0.76	4,216 15,946	3.28
		Mean......	3.59

The fourth column is computed as follows:—

Suppose that the strengths are as the x power of the diameters, then for the first bar we have

$$\left(\frac{1.51}{0.766}\right)^x = \frac{23025}{1948} \text{ or } 1.971^x = 11.30$$

$$\therefore x = \frac{log.\ 11.30}{log.\ 1.971} = 3.57.$$

The others are computed in the same way.

An examination of the table shows that when the square section is the same, the strength varies inversely as some function of the length. Thus, of two bars whose cross section is one square inch, the one five feet long is nearly four times as strong as the one ten feet long.

Let l = length of one,
l' = " of other,
d = diameter of first one,
d' = " of the second one, and
y = the power of the length.

Then the strength of the first one is, $P = \text{constant} \times \frac{d^{3.59}}{l^y}$

" " " second is, $P' = \text{constant} \times \frac{d'^{3.59}}{l'^y}$

$$\therefore \frac{P}{P'}\left(\frac{d'}{d}\right)^{3.59} = \left(\frac{l'}{l}\right)^{y}$$

in which substitute the values from any two experiments. Thus if we take from the table

$l' = 10$ feet, $d' = 1$ inch, $P' = 4225$ lbs., and
$l = 5$ feet, $d = 1$ inch, and $P = 18038$ lbs., we have

$$\frac{18038}{4225} = 2^{y}$$

$$\therefore y = \frac{\log.\ 4.2694}{\log.\ 2} = 2.094$$

Proceed in a similar way with each of the others and take the mean of the results for the power to be used. In this way was formed the following

TABLE

For the absolute strength of columns.

in which P = crushing weights in gross tons,
d = the external diameter, or side of the column in inches,
d_1 = the internal diameter of the hollow in inches, and
l = the length in feet.

Kind of Column.	Both ends rounded, the length of the column exceeding fifteen times its diameter.	Both ends flat, the length of the column exceeding thirty times its diameter.
	TONS.	TONS.
Solid Cylindrical Columns of cast iron..................	$P = 14.9\ \frac{d^{3.76}}{l^{1.7}}$	$P = 44.16\ \frac{d^{3.55}}{l^{1.7}}$
Hollow Cylindrical Columns of cast iron..............	$P = 13\ \frac{d^{3.76} - d_1^{3.76}}{l^{1.7}}$	$P = 44.34\ \frac{d^{3.55} - d_1^{3.55}}{l^{1.7}}$
Solid Cylindrical Columns of wrought iron.............	$P = 42\ \frac{d^{3.76}}{l^2}$	$P = 133.75\ \frac{d^{3.55}}{l^2}$
Solid Square Pillar of Dantzic oak....................	....	$P = 10.95\ \frac{d^4}{l^2}$
Solid Square Pillar of red dry deal......................	....	$P = 7.81\ \frac{d^4}{l^2}$

The above formulas apply only in cases where the length is so great that the column breaks by bending and not by simple crushing. If the column be shorter than that given in the table, and more than four or five times its diameter, the strength is found by the following formula:

$$W = \frac{P.CK}{P + \frac{3}{4}\,CK} \dots\dots\dots\dots\dots\dots (24)$$

in which P = the value given in the preceding table,
K = the transverse section of the column in square inches,
C = the modulus for crushing in tons (gross) per square inch, and
W = the strength of the column in tons (gross).*

Experiments have been made upon steel pillars which gave good results.†

63. WEIGHT OF PILLARS.—From the first formula of the preceding table we find

$$d = \frac{P^{\frac{1}{3.76}}\,l^{\frac{0.85}{1.88}}}{14.9^{\frac{1}{3.76}}}$$

The area of the cross-section is $\frac{1}{4}\,\pi\,d^2$, and the volume in inches $= \frac{12}{4}\,\pi\,d^2\,l$.

Cast iron weighs 450 pounds to the cubic foot, hence the

$$\text{weight} = \frac{450}{1728} \times 3 \times \pi d^2 \times l = \frac{225}{288} \times 3.1416 \times \frac{P^{\frac{1}{1.88}}\,l^{\frac{1.79}{0.94}}}{14.9^{\frac{1}{1.88}}}$$

If P is given in pounds, this coefficient must be divided by $2240^{\frac{1}{1.88}}$.

$\therefore$ weight in pounds $= 0.0121702\,(P^{lbs.}\,l^{3.58})^{\frac{1}{1.88}} \dots (25)$

If the pillar is hollow the section of the iron is $\frac{1}{4}\,\pi\,(d^2 - d_1^2)$, and if n is the ratio of the diameters, so that $d_1 = n\,d$ this becomes

$\frac{1}{4}\,\pi\,d^2\,(1 - n^2)$; and its volume in inches $= \frac{12}{4}\,\pi\,d^2\,(1 - n^2)\,l$;

and its weight in pounds $= \frac{450}{1728} \times 3 \times \pi\,d^2\,(1 - n^2)\,l$.

---

* James B. Francis, C. E., has published a set of tables which gives the strength of cast-iron columns, of given dimensions, by means of equation (24), and also by those in the above table.

† *London Builder*, No. 1211.

If the value of d from the second equation of the first column in the preceding table, be substituted in the preceding equation, we find the

weight in pounds =

$$\frac{25}{32}\frac{\pi}{\left(2240 \times 13\right)^{\frac{1}{1.88}}}\frac{1 - n^2}{\left(1 - n^{3.76}\right)^{\frac{1}{1.88}}} l \left(P.l^{1.7}\right)^{\frac{1}{1.88}} \quad (26)$$

To find the weight for a five-fold security these results must be multiplied by $5^{\frac{1}{1.88}}$.

Proceeding in this way with each of the cases given above and we form the following:

TABLE

Of the weights in pounds of pillars in terms of their lengths (l) *in feet, when loaded to one fifth their crushing strength* (P) *in pounds.*

Kind of Pillar.	Weight in pounds. Both ends rounded. $l > 15\ d$.	Weight in pounds. Both ends flat. $l > 30\ l$.
Solid Cylindrical Column of cast iron.	$0.028648953\ (P.l^{3.58})^{\frac{1}{1.88}}$	$0.009321706\ (P.l^{3.475})^{\frac{1}{1.775}}$
Hollow Cylindrical Columns of cast iron, $d_1 = nd$.	$0.024392078 \frac{1-n^2}{(1-n^{3.76})^{\frac{1}{1.88}}} \times (P.l^{3.58})^{\frac{1}{1.88}}$	$0.009300164 \frac{1-n^2}{(1-n^{3.55})^{\frac{1}{1.775}}} \times (P.l^{3.475})^{\frac{1}{1.775}}$
if $n=0.98$	$0.003881655\ (P.l^{3.58})^{\frac{1}{1.88}}$	$0.001658133\ (P.l^{3.475})^{\frac{1}{1.775}}$
if $n=0.95$	$0.006001775\ (P.l^{3.58})^{\frac{1}{1.88}}$	$0.002489827\ (P.l^{3.475})^{\frac{1}{1.775}}$
if $n=0.925$	$0.007265678\ (P\ l^{3.58})^{\frac{1}{1.88}}$	$0.002987882\ (P.l^{3.475})^{\frac{1}{1.775}}$
if $n=0.90$	$0.008396144\ (P.l^{3.58})^{\frac{1}{1.88}}$	$0.003406063\ (P.l^{3.475})^{\frac{1}{1.775}}$
if $n=0.875$	$0.009373430\ (P.l^{3.58})^{\frac{1}{1.88}}$	$0.003773531\ (P.l^{3.475})^{\frac{1}{1.775}}$
if $n=0.85$	$0.010261387\ (P.l^{3.58})^{\frac{1}{1.88}}$	$0.004106903\ (P.l^{3.475})^{\frac{1}{1.775}}$
if $n=0.80$	$0.011862713\ (P.l^{3.58})^{\frac{1}{1.88}}$	$0.004702651\ (P.l^{3.475})^{\frac{1}{1.775}}$
$n=0.75$	$0.013297905\ (P.l^{3.58})^{\frac{1}{1.88}}$	$0.005233352\ (P.l^{3.475})^{\frac{1}{1.775}}$
Solid Cylindrical Columns of Wrought Iron,	$0.014115831\ (P.l^{3.58})^{\frac{1}{1.88}}$	$0.004993604\ (P.l^{3.775})^{\frac{1}{1.775}}$
Square Column of Danticz Oak.	(Cubic foot weighs 47.24 pounds.)	$0.001223770\ (P^{\frac{1}{2}}l^2)$

If the thickness of the metal (t) and the external diameter are given, n may be found as follows: $d - 2t$ = internal diameter, hence $n = \frac{d-2t}{d} = 1 - \frac{2t}{d}$. For instance, if the external diameter is 6 inches, and the thickness $\frac{3}{8}$ of an inch, the internal diameter is $5\frac{1}{4}$ inches and $n = \frac{5\frac{1}{4}}{6} = 0.875$.

The iron used in the preceding experiments was Low Moor No. 2, whose strength in columns is about the mean of a great variety of English cast iron, the range being about 15 per cent. above and below the values given above.

64.—CONDITION OF THE CASTING.—Slight inequalities in the thickness of the castings for pillars does not materially affect the strength, for, as was found by Mr. Hodgkinson, thin castings are much harder than thicker ones, and resist a greater crushing force. In one experiment the shell of a hollow column resisted about 60 *per cent.* more per square inch than a solid one.* But the excess or deficiency of thickness should not in any case exceed 25 per cent. of the average thickness.† Thus, if the average thickness is one inch, the thickest part should not exceed $1\frac{1}{4}$ inch, and the thinnest part should not be less than $\frac{3}{4}$ of an inch.

It is also found that in large castings the crushing strength of the part near the surface does not much exceed that of the internal parts.

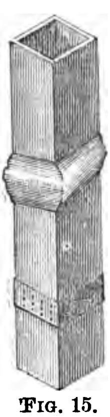


FIG. 15.

FIG. 16.

65. COMPRESSION OF TUBES.—BUCKLING.—Wrought iron tubes when subjected to longitudinal compressive stresses may yield by crushing like a block, or by bending like a beam, or by buckling. The first takes place when the tube is very short; the second, when it is long compared with the diameter of the tube; and the last, for some length which it is difficult to assign, intermediate between the others.

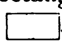
The appearance of a tube after it has yielded to buckling is shown in Figs. 15 and 16.

* *Phil. Trans.*, 1857, p. 890. † *Stoney on Strains*, vol. ii., p. 206.

The experiments heretofore made do not indicate a specific law of resistance to buckling; but the following general facts appear to be established:*—

1. The resistance to buckling is always less than that to crushing; and is nearly independent of the length.

2. Cylindrical tubes are strongest; and next in order are square tubes, and then rectangular ones.

3. Rectangular tubes, , are not as strong as tubes of this form . The tubes in bridges and ships are generally rectangular or square.

COLLAPSE OF TUBES.

66. THE RUPTURE OF TUBES which are subjected to great external normal pressure is called "a collapse." The flues of a steam-boiler are subjected to such an external pressure, and in view of the extensive use of steam power, the subject is very important. The true laws of resistance to collapsing were unknown until the subject was investigated by Wm. Fairbairn. Experiments were carefully made, and the results discussed by him with that scientific ability for which he is so noted. They were published in the Transactions of the Royal Society, 1858, and republished in his "Useful Information for Engineers," second series, page 1.

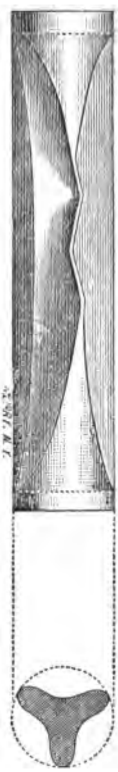


FIG. 17.

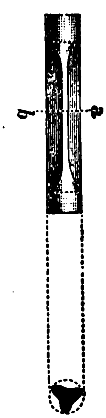


FIG. 18.

The tubes were closed at each end and placed in a strong cylindrical vessel made for the purpose, into which water was forced by a hydraulic press, thus enabling him to cause any desirable pressure upon the outside of the tube. In order to place the tube as nearly as possible in the condition of

* *Civ. Eng. and Arch. Jour.*, vol. xxviii., p. 28.

a flue in a steam-boiler, a pipe which communicated with the external air was inserted into one end of the tube. This pipe permitted the air to escape from the tube during collapse.

The vessel, pipe, tube, and their connections were made practically water-tight, and the pressure indicated by gauges.

Fig. 17 shows the appearance and cross-section at the middle of the short tubes after the collapse; and Fig. 18 of a long one. Although no two tubes appeared exactly alike after the collapse, yet the examples which I have selected are good types of the appearances of thirty tubes used in the experiments.

The tubes in all cases collapsed suddenly, causing a loud report. In the first and second tubes the ends were supported by a rigid rod, so as to prevent their approaching each other when the sides were compressed.

The following tables give the results of the experiments:—

TABLE I.

Mark.	No.	Thickness of Plate, inches. *t*.	Diameter in inches. *d*.	Length in inches. *L*.	Pressure of Collapse, lbs. pr. sq. in. of Surface. *P*.	Product of Pressure and Length. *P. L.*	Product of the Pressure, Length, and Diameter. *P. L. d.* = *p*.
A	1	0.043	4	19	170	3230	
B	2	"	"	19	187	2603	10412
C	3	"	"	40	65	2600	10400
D	4	"	"	38	65	2470	9880
E	5	"	"	60	43	2580	10320
F	6	"	"	60	140*	2800	
					Mean	2714	10253
G	7	"	6	30	48†	1440	
H	8	"	"	29	47†	1263	
J	9	"	"	59	32	1888	11328
K	10	"	"	30	52	1560	9360
L	11	"	"	30	65	1950	11700
M	12	"	"	30	85‡	?	
					Mean	1620	10796
N	13	"	8	30	39	1170	9360
O	14	"	"	39	32	1248	9984
P	15	"	"	40	31	1240	9920
					Mean	1219	9754
Q	16	"	10	50	19	950	9500
R	17	"	"	30	33	990	9900
					Mean	970	9700
S	18	"	12.2	58½	11.0	643.7	7850
T	19	"	12	60	12.5	750	9000
V	20	"	"	30	22	662	7920
					Mean	685.2	8256

* This tube had two solid rings soldered to it, 20 inches apart, thus practically reducing it to three tubes, as shown in Fig. 19.

FIG. 19.

† The ends of both were fractured, causing collapse, perhaps before the outer shell had attained its maximum.

‡ A tin ring had been left in by mistake, thus causing increased resistance to collapsing.

67. DISCUSSION OF RESULTS.—By comparing the tubes of the same diameter and thickness, but of different lengths, we see that the long tubes resist less than the short ones; hence, the strength is an *inverse function* of the length, and an examination of the seventh column shows that it is nearly a simple inverse function of the length. The first of the 4 inch tubes is so much stronger than the others, it may be neglected in determining the *law of resistance*, although it differs from a mean of all the others by less than $\frac{1}{8}$ of the mean. An examination of the several cases indicates that we may *safely* assume that *the resistance to collapsing varies inversely as the lengths of the tubes.**

The mean of the results for the several diameters in the last column shows that the resistance diminishes somewhat more rapidly than the diameter increases; but this includes the error, if any, of the preceding hypothesis. As the power of the diameter is but little more than unity, it seems safer to conclude, for all tubes less than 12 inches in diameter, as Fairbairn does, that *the resistance of tubes to collapsing varies inversely as their diameters.*

68. LAW OF THICKNESS.—Experiments were also made to determine the law of resistance in respect to the thickness. Comparatively few experiments were made of this character, but these few gave remarkably uniform results. One of the

---

* A more exact law may be found as follows:—Let $P =$ the compressing force per square inch; C = a constant for any particular diameter and thickness, $l =$ the length, and n the unknown power. Then

$$P = \frac{C}{l^n} \text{ for one case.}$$

$$P_1 = \frac{C}{l_1^n} \text{ for another.}$$

$$\therefore n = \frac{\log. \frac{P}{P_1}}{\log. \frac{l_1}{l}}$$

By means of this equation, and any two experiments in which the thickness and diameter are the same, n may be found, and by using several experiments a series of values may be found from which the most probable result can be obtained. But in this case the *mean* result is so near unity, there is no practical advantage secured by finding it.

tubes (No. 24), was made with a butt joint, as shown in Fig. 20, and the others with lap joints, as in Fig. 21.

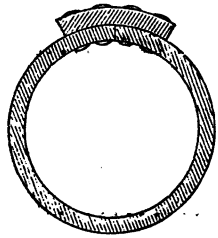


FIG. 20.

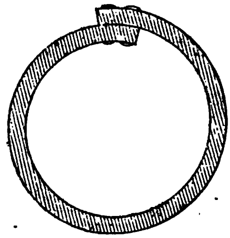


FIG. 21.

The following are the results of the experiments:—

TABLE II.

Mark.	No.	Thickness. t.	Diameter. d.	Length in inches. L.	Pressure per square inch. P.	$P\ L$.	Product. $L.\ P.\ d. = p$.
W	21	0.25	9	37	(450)	Uncollapsed.	
X	22	0.25	18¾	61	420	25620	480375
Y	23	0.14	9	37	262	9694	89046
Z	24	0.14	9	37	378	13986	125874
JJ	33	0.125	14½	60	125	7500	108750

Tubes Nos. 23 and 24 were exactly alike in every respect except their joints; and it appears that the butt joint, No. 24, is 1.41 times as strong as the lap joint, a gain of 41 per cent. But this is a larger gain than is indicated in other cases; for instance, No. 33, which is also a lap joint, offers a greater resistance as indicated in the last column, than No. 23, although the former is not as thick as the latter. Still it seems evident that butt joints are stronger than lap joints, for with the former the tubes can be made circular, and there is no cross strain on the rivets, conditions which are not realized in the latter.

The resistance of the 23d is so small compared with others, it is rejected in the analysis.

We observe that the resistance varies as some power of the thickness; if then C and n be two constants to be determined by experiment, and we use the notation given above, we shall have for the pressure of collapse of one tube,

$$P = \frac{Ct^n}{dL} \therefore dPL = p = Ct^n \ldots\ldots\ldots\ldots (27)$$

and for another tube

$$P_1 = \frac{Ct_1^n}{d_1L_1} \therefore d_1 P_1 L_1 = p_1 = Ct_1^n \ldots\ldots (28)$$

Hence we have

$$\frac{p}{p_1} = \left(\frac{t}{t_1}\right)^n$$

$$\text{or, } n = \frac{\log. p - \log. p_1}{\log. t - \log. t_1} \ldots\ldots\ldots\ldots (29)$$

$$\text{and } C = \frac{p}{t^n} = \frac{p_1}{t_1^n} \ldots\ldots\ldots\ldots (30)$$

TO FIND THE CONSTANTS n AND C.

The mean of the mean of the values of p from Table I. is $p = \frac{1}{5}$ [10253 + 10796 + 9754 + 9700 + 8256] = 9752 and $t = 0.043$.

Using these values and others taken from the preceding tables, and the following values may be found for n:—

In equation (29) make $p = 480375$, $t = 0.25$, $p_1 = 9752$, $t_1 = 0.043$; and we get

$$n = \frac{\log. 480375 - \log. 9752}{\log. 0.25 - \log. 0.043} = 2.200.$$

Similarly, taking $p = 480375$, $t = 0.25$, $p_1 = 10253$, $t_1 = 0.043$; and we get

$$n = \frac{\log. 480375 - \log. 10253}{\log. 0.25 - \log. 0.043} = 2.185.$$

The mean value of p for all but the 12-inch tubes in Table I. is

$$p = \tfrac{1}{4}(10253 + 10796 + 9754 + 9700) = 10125;$$

hence, using $p = 125874$, $t = 0.14$, $p_1 = 10125$, $t_1 = 0.043$; and we get

$$n = \frac{\log. 125874 - \log. 10125}{\log. 0.14 - \log. 0.043} = 2.134;$$

and taking $p = 108750$, $t = 0.125$, $p_1 = 10125$ and $t_1 = 0.043$; we get

$$n = \frac{\log. 108750 - \log. 10125}{\log. 0.125 - \log. 0.043} = 2.203,$$

and the mean of these results is, $n = 2.18$.

Fairbairn made it 2.19 by including some data which I have rejected as paradoxical; I have also given more weight to those cases which gave nearly uniform results. The difference, however, of 0.01 is too small to seriously affect practical results.

To determine the constant, C, substitute the proper values taken from the preceding tables in equation (30), and we have for four cases the following:—

$$C = \frac{9752}{0.043^{2.18}} = 9,298,900.$$

$$C = \frac{480375}{0.25^{2.18}} = 9,864,300.$$

$$C = \frac{125874}{0.14^{2.18}} = 9,144,000.$$

$$C = \frac{108750}{0.125^{2.18}} = 10,109,400.$$

The mean of which is $C = 9,604,150$. Calling $C =$ 9,600,000 and equation (27) becomes:

$$P = 9,600,000 \frac{t^{2.18}}{d.L} \text{........................} (31)$$

If L be given in feet, so that $L = 12\ L_f$, we have

$$P = 800,000 \frac{t^{2.18}}{d.L_f} \text{........................} (32)$$

The coefficient, 9,600,000, applies only to the kind of iron used; but the exponent, 2.18, is supposed to be constant for all kinds of iron.

69. FORMULA FOR THICKNESS TO RESIST COLLAPSING. —Equation (31) readily gives the following expression *for finding the thickness in inches of a tube to resist collapsing*:—

$$\log.\ 100\ t = \frac{\log.\ P + \log.\ (d.\ L)}{2.18} - 1.203 \text{....} (33)$$

70. ELLIPTICAL TUBES.—Experiments made upon *elliptical tubes* showed that the preceding formula would give the strength, if the diameter of the circle of curvature at the extremity of the minor axis is substituted for *d*. The diameter of curvature is $\frac{2a^2}{b}$, in which a is the major and b the minor axis.

Experiments made upon tubes in which the ends were not connected by internal rods, showed that the resistance was inversely as their length.

71. VERY LONG TUBES.—Some experiments were made upon a tube 35 feet long and one 25 feet long. Sufficient pressure was applied to distort them, but not to collapse them, and it was found that Equation (31) erred by at least 20 per cent., giving too small an amount. It was, however, very evident that the length was still a very important element in the strength.

72. COMPARISON OF STRENGTH FROM EXTERNAL AND INTERNAL PRESSURE.—Let p be the internal pressure per square inch at which the tube is ruptured, then for tubes of the same thickness and diameter we have from Equations (18) and (32), by calling $T = 30{,}000$ lbs.,

$$\frac{p}{P} = \frac{1}{13.33} \frac{L.}{t^{1.18}}$$

If $p = P$, then $L = 13.33\ t^{1.18}$.

If $t = 0.25$, then we find $L = 2.59$ feet, that is, a tube whose thickness is $\frac{1}{4}$ of an inch, and whose length is 2.59 feet, is equally strong whether subjected to internal or external pressure.

If the tube is so thick that the unequal stretching of the fibres must be considered, then Equation (20) must be compared with Equation (32), in which case we have:—

$$\frac{p}{P} = \frac{T}{800{,}000} \times \frac{d.\ L_f}{(r + t)\ t^{1.18}}$$

If $p = P$, $T = 40{,}000$ lbs., and $2r = d = 4$ inches; then $2t^{1.18} + t^{2.18} = \frac{1}{5} L_f$.

If $t = \frac{1}{2}$ inch, $L = 5.504$ feet.

If $t = 1$ " $L = 15.000$ feet.

73. RESISTANCE OF GLASS GLOBES TO COLLAPSING.—Fairbairn also determined that glass globes and cylinders followed the same *general law* of resistance. For globes of flint glass he found:

$$P_1 = 28,300,000 \frac{t^{1.4}}{d^{3.4}} \ldots\ldots (34)$$

and for cylinders of flint glass:

$$P_1 = 740,000 \frac{t^{1.4}}{d\,L} \ldots\ldots (35)$$

providing that their length is not less than twice, nor more than six times their diameter. Dividing Equation (35) by (31) gives $\frac{P_1}{P} = \frac{0.0770}{t^{0.78}}$.

If $t = 0.043$ in., $\frac{P_1}{P} = 0.896$; or the glass cylinder is nearly $\frac{9}{10}$ as strong as the iron one. If they are equally strong, $P = P_1$ $\therefore t = 0.0373$ of an inch.

CHAPTER III.

THEORIES OF FLEXURE AND RUPTURE FROM TRANSVERSE STRESS.

74.—REMARK.—The ancients seem to have been entirely ignorant of the laws which pertain to the resistance of solid bodies. They made some rude experiments to determine the absolute strength of some solids, especially of stone. They may have recognized some general facts in regard to the strength of beams, such as that a beam is stronger with its broad side vertical than with its narrow side vertical, but we find no trace of any law which was recognized by them. This department of science belongs wholly to modern times. A very brief sketch of the history of its development is given below.*

The greater part of this chapter will be much better understood after reading Chapters. IV, V, and VI.

75.—GALILEO'S THEORY.—Galileo was the first writer, of whom we have any knowledge, who endeavored to establish the mathematical laws which govern the strength of beams.† He assumed—

1st. That none of the fibres were elongated or compressed.

2d. When a beam is fixed at one end, and loaded at the other, it breaks by turning about its lower edge, Fig. 22, or if it be supported at its ends and loaded at the middle of the length, it would turn about the upper edge; hence every fibre would resist tension.

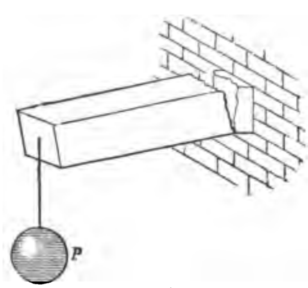


FIG. 22.

3. Every fibre acts with equal energy. From these he readily

* For a more complete history, see introduction to "*Résistance des Corps Solides*," par Navier. 3d edition. Paris, 1864.

† *Opere di Galileo.* Bologne, 1656.

deduced,—that, when one end is firmly fixed in a wall or other immovable mass, the *moment of resistance of the section* is equal to the sum of all the fibres, or the transverse section, multiplied by the resistance of a unit of section, multiplied by the distance of the centre of gravity from the lower edge. Hence, in a rectangular beam, if

T = the tenacity of the material,
b = the breadth, and
d = the depth of the beam;

the moment of resistance is

$$Tbd \times \tfrac{1}{2}d = \tfrac{1}{2}\, Tbd^2 \qquad (34)$$

76.—ROBERT HOOKE'S THEORY.—Robert Hooke was one of the first, and probably the first, to recognize the compressibility of *solids* when under pressure. In 1678 he announced his famous principle, *Ut tensio sic vis;* which he gave in an anagram in 1676, and stated as the basis of the theory of elasticity that the *extensions* or *contractions* were proportional to the forces which produce them, and also that when a bar was bent the material was compressed on the concave side and extended on the convex side.

77.—MARIOTTE'S AND LEIBNITZ'S THEORY.—Mariotte, in 1680, investigated the subject, and finally stated the following principles:—

1st. The material is extended on the convex side and compressed on the concave side.

2d. In *solid rectangular* sections the line of invariable fibres (or *neutral axis*) is at half the depth of the section.

3d. The elongations or compressions increase as their distance from the neutral axis.

4th. The resistance is the same whether the neutral axis is at the middle of the depth or at any other point.

5th. The lever arm of the resistance is $\frac{2}{3}$ of the depth.

We here find some of the essential principles of the resistance to flexure, as recognized at the present day; but the two last are erroneous. As hereafter shown, the neutral axis is at half the depth, and the lever arm is $\frac{2}{3}$ of $\frac{1}{2}$ the depth.

Leibnitz's theory, given in 1684, was the same as Mariotte's.

78.—JAMES BERNOUILLI'S THEORY was essentially the same as Mariotte's, except that he stated that extensions and compressions were not proportional to the stresses. "For," said he, "if it is true, a bar might be compressed to nothing with a finite force." On this point see Article 16. He was the first to give a correct expression for the equation of the elastic curve.

79. PARENT'S THEORY.—Parent, a French academician of great merit, but of comparatively little renown, published, in 1713, as the result of his labors, the following principles, in addition to those of his predecessors:—

1st. The total *resistance* of the compressed fibres equals the total *resistance* of the extended fibres.

2d. The *origin of the moments* of resistance should be on the neutral axis.

By the former of these principles the position of the neutral axis may be found, when the straining force is normal to the axis of the beam; and by the latter he corrected the error of Mariotte and Leibnitz; showing that the ratio of the *absolute* to the *relative* strength is as *six* times the length to the depth instead of three, as will be shown hereafter.

80. COULOMB, IN 1773, PUBLISHED the most scientific work on the subject of the stability of structures which had appeared up to his time. He deduced his principles from the fundamental equations of statics, and generalized the first of the principles of Parent, which is given above, by saying that *the algebraic sum of all the forces must be zero on the three rectangular axes.* This establishes the position of the neutral axis when the applied forces are oblique to it, as well as when they are normal. He also remarked, that if the proportionality of the compressions and extensions do not remain to the last, or to the point of rupture, the final neutral axis will not be at the centre of the section.

81. MODULUS OF ELASTICITY.—In 1807 Thomas Young introduced the term *modulus of elasticity*, which we have defined as the coefficient of elasticity in Article 5. After this several writers, among them Duhamel, Navier in his early writings, and Barlow in his first work, stated the erroneous principle, *that the sum of the* MOMENTS *of the resistances to compression equalled those for tension.*

82. IN 1824 NAVIER PUBLISHED the lectures which he had given to *l'École des Ponts et Chaussées*, in which he established more clearly those principles of elastic resistance, and resistance to rupture, which have since his day been accepted by nearly all writers. He was the first to show that when the stress is perpendicular to the axis of the beam, *the neutral axis passes through the centre of gravity of the transverse sections.* His most important modifications in the *analysis* was in making $ds = dx$, or otherwise, considering that *for small deflections the tangent of the angle which the neutral axis makes with the original axis of the beam is so small compared with unity that it may be neglected;* and also, that the lever arm of the force remains constant during flexure. These principles we have used in Chapter V. He resolved many problems not before attempted, and became an eminent author in this department of science.

83. THE COMMON THEORY.—The theories of flexure and of rupture which result from these numerous investigations, I will call, for convenience, the *common theory.* It consists of the following hypotheses:—

1st. The fibres on the convex side are extended, and on the concave side are compressed, and there are no strains but compression and extension.

2d. Between the extended and compressed fibres (or elements) there is a surface which is neither extended nor compressed, but retains its original length, and which is called the neutral surface, or in reference to a plane of fibres it is called the *neutral axis.*

3d. The strains are proportional to their distance from the neutral axis.

4th. The transverse sections which were normal to the neutral axis of the beam before flexure, remain normal to the neutral axis during flexure.

5th. A beam will rupture either by compression or extension when the *modulus of rupture* is reached.

6th. The *modulus of rupture* is the strain at the instant of rupture upon a unit of the section which is most remote from the neutral axis on the side which first ruptures. This is called R.

It is found that this theory does not conform well with the results of experiment.* For instance, if a cast-iron beam be supported at its ends, and broken by a weight placed at the middle, it appears from the theory above given that the beam would break when the strain (R) on the extreme fibres equals the value of the tenacity (T) of the metal—or 16,000 lbs. (See page 58.) But the value of R as found from the formula, $R = \frac{3}{2}\frac{Pl_1}{bd^2}$ which is deduced in accordance with the above theory, and is given in Chapter VI., is about 35,000 lbs. (See the table in the Appendix.) This value is less than the crushing strength, C, of the metal—or 96,000 lbs. (See page 58). Hence the value of R is nearly $2\frac{1}{4}$ times that of T, and more than $\frac{1}{3}$ that of C.

Again, we have for Ash

$$T = 17{,}200 \text{ pounds};$$
$$C = 9{,}000 \quad \text{“} \quad ;$$
$$R = 12{,}000 \quad \text{“} \quad ;$$
$$\therefore R = 1\tfrac{1}{3}\, C \text{ and } \tfrac{2}{3}\, T \text{ nearly}.$$

* *Mosley's Mech. and Arch.*, p. 557. "The elasticity of the material has been supposed to be perfect up to the instant of rupture, but the extreme fibres are strained much beyond their elastic limits before rupture takes place, while the fibres near the neutral axis are but slightly strained, and hence the law of proportionality is not maintained, and the position of the neutral axis is changed, and the sum of the moments is not accurately $\frac{RI}{d_1}$ (see equation 170). To determine the influence of these modifications we must fall back upon experiment, and it has been found *in the case of rectangular beams* that the error will be corrected if we take $\frac{1}{m}\, T\, (= R)$ instead of T, where m is a constant depending upon the material."

Weisbach, vol. ii., 4th ed., p. 68, foot-note, says, "Excepting as exhibiting approximately the laws of the phenomena, the theory of the strength of materials has many practical defects."

Maj. Wade, in his Report to the Ordnance Department, p. 1, says:—"A trial was made with cylindrical bars in place of square ones. These generally broke at a point distant from that pressed, and the results were so anomalous that the use of them was soon abandoned. The formula by which the strength of round bars is computed appears to be not quite correct, for the unit of strength in the round bars is uniformly much higher than in the square bars cast from the same iron."

A similar result is found for other materials. Hence generally, the value of R for any given material is between those of T and C, but there is no known relation between them which would enable us to determine the value of one from the other two. The values of R in the tables were deduced from experiments upon rectangular beams, as will hereafter be shown; and hence, if the common theory is correct, R should equal the value of the lesser resistance, whether it be for compression or extension; but it does not. This discrepancy between theory and the results of experiment, led Mr. Barlow to investigate the subject farther, and it resulted in a new theory which he calls "Resistance to Flexure"—an expression which I consider unfortunate, as it does not express his idea. "Longitudinal Shearing" would express his idea better, as will appear from the following article:—

84. BARLOW'S THEORY.—According to the common theory the resistance at a section is the same as if the fibres acted independently of each other, and the transverse section remained normal to the neutral axis. But Barlow supposed that in order to keep the transverse sections normal to the neutral axis, the consecutive longitudinal planes of fibres must *slide* over each other, and to this movement they offer a resistance. (This point is discussed in Chapter IX.)

He presented his view to the Royal Society (Eng.), in 1855, and it has since been published in the *Civil Engineer and Architect's Journal*, vol. xix., p. 9, and vol. xxi., p. 111.* The subject is there discussed in a very able and thorough manner, and although he may have failed to establish his theory, yet the results of his analysis seem to agree more nearly with the results of experiment than those obtained by any other theory heretofore proposed.

It is admitted in this theory that a beam will rupture when the stress upon any fibre equals its tenacity, or its resistance to compression, as the case may be. But, on the other hand, when the adjacent fibres are unequally strained, as they are in the case of flexure, it requires a greater stress to produce this

* *Civ. Eng. and Arch. Jour.*, vol. xix., p. 9, Barlow says that the strength of a cast-iron rectangular bar, as found from existing theory, cannot be recon-

strain than it would if the fibres acted independently, according to the previously assumed law. This, Barlow makes evident from the following example:—

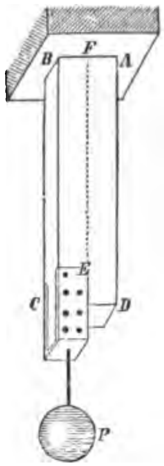


FIG. 23.

If a weight P, Fig. 23, is suspended on a prismatic bar, $BCEF$, all the fibres will be equally strained, and hence equally elongated.

But if the bar $ABCD$ be substituted for the former, and the weight P acts upon a part of the section, as shown in the figure, it is evident that all the fibres will not be equally strained, and hence will not be equally elongated; and if the force P was just sufficient to rupture the bar $FBCE$, it will not be sufficient to rupture the bar $ABCD$, although P acts directly upon the same section, for the cohesion of the particles along FE will not permit the fibres next to that line to be elongated as much as if the part $AFED$ were removed; and these fibres will act upon those adjacent, and so on, until they produce an effect upon BC. From this we see that it takes a greater weight than P acting upon the section EC to produce a strain T per unit of section, when the part $ADEF$ is added. It is also evident that if the section of $ABCD$ is twice as great as $FBCE$, it will not take twice P to rupture the fibres on the side BC.

A phenomenon similar to this takes place in transverse strain. One side is compressed and the other elongated; and the fibres less strained aid those which are more strained by virtue of the cohesion which exists between them, and it takes a greater load to cause a strain, T, longitudinally upon the fibres on the convex side, or of C upon those on the concave side, than it would if there were no cohesion between the horizontal laminæ.

ciled with the results of experiment if the neutral axis be at the centre of the sections. He then proceeded to show by experiment that the neutral axis is at the centre, and then remarked that the formula commonly used for a beam supported at the ends and loaded in the middle, or $P = \frac{2}{3}\frac{T'bd^2}{l}$ did not give half the actual strength if T' is the tenacity of the iron. He then proceeds to point out a new element of strength, which he calls "Resistance to Flexure."

This may be illustrated by a pile of boards, Fig. 24. Sup-

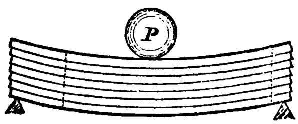


FIG. 24.

pose that the boards are very thin and perfectly smooth. When they are bent they will retain their original length, and will project past each other as shown in the figure. Also if before they were deflected straight lines were drawn with a pencil or otherwise perpendicularly across the pile, and then the whole deflected, it will be found that the lines will not remain continuous but will be broken. If now there be considerable friction between the boards, those on the concave side will be compressed, and those on the convex side will be elongated; and the cross lines will be more nearly continuous than before. Still more, if the successive layers be infinitely thin and held together by cohesion, the elements on the concave side will be still more compressed and those on the convex more extended than they were in the former case, and the cross lines will remain straight and normal to the neutral axis, as shown in Fig. 25.

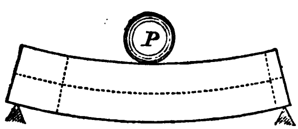


FIG. 25.

There is, then, at the time of the rupture of a beam, according to this theory, a tensile strain on the extended fibres, and a compressive strain on the other fibres, and a longitudinal shearing strain between the fibres, due to cohesion. We think, however, that too much importance is given to the longitudinal shearing as an *element of strength*.

Barlow's Theory consists of the following hypotheses:—

1st. The fibres or elements on the convex side are extended, and on the concave side compressed.

2d. There is a neutral surface, as in the common theory.

3d. The tensile and compressive strains on the fibres are proportional to their distances from the neutral axis.

4th. That in addition to these there is a "Resistance to flexure" or longitudinal shearing strain, which consists of the following principles:—

a. It is a strain in addition to the direct extensive and compressive forces, and is due to the lateral cohesion of the adjacent surfaces of fibres or particles, and to the elastic reaction which ensues when they are unequally strained.

b. It is evenly distributed over the surface, and consequently within the limits of its operation its centre of action will be at the centre of gravity of the compressed or of the extended section. This force for solid beams Barlow calls ϕ, and for **T** or **I** sections, or open-built beams, it is easily deduced from the following principle:—

c. It is proportional to and varies with the *inequality* of strain between the fibres nearest the neutral axis and those most remote.

From this it appears that if d' is the depth of the horizontal flanges of the **I** section, and d_1 the distance of the most remote fibre from the neutral axis, then the *resistance to flexure of the flanges* will be $\phi \frac{d'}{d_1}$ and similarly for other forms.

5. Sections remain normal to the neutral axis during flexure.

6. Rupture of solid beams takes place when the strain on a unit of section is $T + \phi$, or $C + \phi$, whichever is smaller, or rather, whichever value is first reached.

Prof. Barlow made no effort to show the value of the *elastic resistance* of longitudinal shearing in a beam under flexure. The effect of this resistance in the flexure of beams will be noticed hereafter.

85. TRANSVERSE ELASTICITY.—If a beam were destitute of elasticity it could not be bent. If it had longitudinal elasticity only, it could be bent by causing the fibres on the convex side to be elongated and those on the concave side to be shortened, as explained in the previous articles. If it had no longitudinal elasticity, but a transverse elasticity, it could be bent by forcing the successive material sections past each other. Let AB, Fig. 26, represent a beam which is supported at its ends, and which

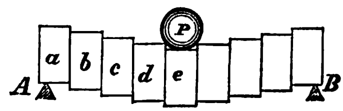


FIG. 26.

is supposed to consist of a succession of perfectly nonelastic parts, as a, b, c, etc., and that these parts are joined by infinitely thin elastic pieces. If a weight P be placed upon the beam, it will cause a deflection similar to that shown in the figure, excepting that the visible effect is greatly exaggerated; but the successive sections will set past each other a small amount. If now we suppose that the transverse elasticity is uniform and continuous from end to end, it seems evident that the deflection will take the form of that shown in Fig. 27, in which the effect

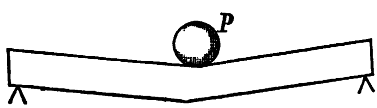


FIG. 27.

due to the elongation and compression of the fibres is supposed to be entirely eliminated. In this case the upper and lower sides are straight from the centre to the ends, but they form an angle with each other at the centre.

It will be shown in Chapter IX, that for amorphous bodies (called isotropes), in which the elasticity is the same in all directions, that the coefficient of the transverse elasticity is $\frac{2}{5}$ of the coefficient of longitudinal elasticity, or $G = \frac{2}{5} E$. Such bodies, however, are more ideal than real. The elasticity is generally different in different directions.*

* See pp. 16 and 17.

86. REMARKS UPON THE THEORIES.—For scientific purposes it is desirable to determine the correct theory of the strength of beams, but the phenomena are so complex that it is not probable that a single general theory can be found which will be applicable to all the irregular forms of beams used in practice. Although Barlow's theory appears plausible, yet according to principle *c* the *resistance to flexure*, ϕ, cannot be uniform over the surface, as stated in principle *b*, because the proportionality of the elongations and compressions do not continue up to the point of rupture. The common theory is faulty beyond what has already been said, in the I section; for in the upper and lower portions the strains on all the fibres are not proportional to their distances from the neutral axis, to realize which the material should be continuous from the neutral axis to the remotest fibres. And Barlow's theory is defective in the same case, on account of the peculiar strains upon the fibres at the angles where the parts join. For rupture, then, we can use these theories to ascertain general facts, and make the results safe in practice by using a proper coefficient of safety; but for flexure the common theory is approximately exact if the elastic limit is not passed, and this is fortunate, for the conditions of stability should be founded upon the elastic properties rather than on the ultimate strength of the material. For the rupture of rectangular beams the common theory will be sufficiently exact if the value of R is used instead of T or C in the formulas.

POSITION OF THE NEUTRAL AXIS.

87. POSITION FOUND EXPERIMENTALLY.—According to Galileo's, Mariotte's, and Leibnitz's theories, the neutral axis is on the surface opposite the side of rupture.

Professor Barlow made the following experiments:—He took a cast-iron beam and drilled holes in its sides, into which were fitted iron pins. He carefully measured the distance between the pins, before and after flexure, by means of a micrometer, and thus found that *in solid cast-iron beams bent by a normal pressure the neutral axis passes through the centre of the sections* (*Civ. Eng. and Arch. Jour.*, vol. xix., p. 10). He also made the same kind of an experiment on a solid rectangular wrought-iron beam, and with the same result (*Civ. Eng. and Arch. Jour.*, vol. xxi., p. 115).

Some years previous to the preceding experiments, he took a bar of *malleable iron* and cut a transverse groove in one side, into which he nicely fitted a rectangular key. When it was bent, the fibres on the concave side were compressed, and the groove made narrower, so that the key would no longer pass through, and thus he showed that the neutral axis was between $\frac{1}{3}$ and $\frac{1}{5}$ the depth of the beam from the compressed side (Barlow's *Strength of Materials*, p. 330; *Jour. Frank. Inst.*, vol. xvi., 2d series, p. 194).

Experiments made at the *Conservatoire des Arts et Métiers*, in 1856, on double T sections, show that it passes through the centre of the sections (Morin, *Résistance des Matériaux*, p. 137). And experiments made at the same time on rectangular wooden beams showed that it passed at or very near the centre of gravity of the sections.

In these experiments the elasticity of the material was not seriously damaged by the strains. To render them complete, the strains should have been carried as near to the point of rupture as possible.

Louis Nickerson, C. E., of St. Louis, made some experiments upon glass by means of polarized light, from which he deduced the following as applicable to that and similar amorphous bodies:—

The neutral axis—as exhibited by polarized light,—from the cohesion of material or other cause is extended to a breadth, and cannot become a true line until, in reference to the cohesion, the tensile and compressive forces are infinite. Also that its longitudinal direction, like the direction of lines of strain, is not an arbitrary one, but resultant from the relative qualities and quantities of all the forces in the beam—its evident place in physics being that of still water between opposing eddies or vortices.

Results obtained showed that the neutral axis is a flexible line, or plane, truly parallel to the top and bottom sides of the rectangular beam and passing through the centres of gravity of its sections *only* when the load is evenly distributed from end to end, or when the beam is infinitely long, and that, when there is a local pressure, the neutral axis is more or less governed in its direction and form by the strain passing from the point of local pressure towards the points of support.

88. POSITION DETERMINED ANALYTICALLY.—We know from statics that the algebraic sum of all the forces on each of the rectangular axes must be zero for equilibrium; hence, if the deflecting forces are normal to the axis of the beam, *the sum of the resistances to compression must equal those for tension.*

1st. Suppose that the coefficient of elasticity for compression equals that for tension. Then will the compressions and extensions be equal at equal distances from the neutral axis. In Fig. 28, let R_c be the strain on a unit of fibres most remote from the neutral axis on the compressed side, and d_c = the distance of the most remote fibre on the same side; then,

$$\frac{R_c}{d_c} = s = \text{strain at a unit's distance from the neutral axis.}$$

Let k_1, k_2, k_3, &c., be the sections of fibres on one side of the neutral axis, at distances of y_1, y_2, y_3, &c., from the axis, and k', k'', k''', &c., and y', y'', y''', &c., corresponding quantities on the other side.

$$\text{Then } s\,(k_1y_1 + k_2y_2 + k_3y_3 + \text{\&c.}) = s\,(k'y' + k''y'' + k'''y''' + \text{\&c.}),$$

$$\text{or, } k_1y_1 + k_2y_2 + k_3y_3 + \text{\&c.} - (k'y' + k''y'' + k'''y''' + \text{\&c.}) = 0,$$

$$\text{or, } \Sigma ky = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (35)$$

or the neutral axis passes through the centre of gravity of the sections.*

If the resistance to compression is greater than for tension, the neutral axis will be nearer the compressed side than when they are equal.

2. Suppose that the coefficient of elasticity is not the same for tension as for compression.

---

* The analytical expression for the ordinate to the centre of gravity is

$$\overline{Y} = \frac{k_1y_1 + k_2y_2 + \text{\&c. } k'y' + k''y'' + \text{\&c.},}{k_1 + k_2 + \text{\&c.} + k' + k'' + \text{\&c.}} \quad \text{or } \overline{Y} = \frac{\iint y\,dy\,dx.}{\int y\,dx}$$

$$\text{If } \iint y\,dy\,dx = 0,\ \overline{Y} = 0.$$

Let Fig. 28 represent the beam. Suppose that the sections CM and EF were parallel before deflection. If through N, the point where EF intersects the neutral axis, KH is drawn parallel to CM, the ordinates between EF and KH will represent the elongations on one side, and the compressions on the other, for those fibres whose original length was LN.

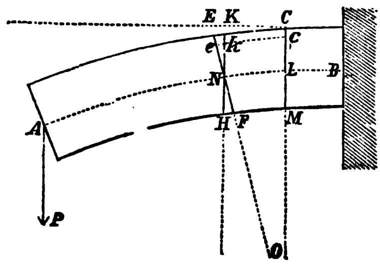


FIG. 28.

Let $l = LN$,

$\lambda = ke =$ the elongation of a fibre at k;

$p =$ a pulling or pushing force which would produce λ;

$y = Nk =$ distance of any fibre from the neutral axis;

$k =$ section of any fibre;

$E_t =$ coefficient of elasticity for tension; and

$E_c =$ " " " compression.

From equation (3) we have,

$$p = \frac{E_t k \lambda}{l} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (36)$$

But λ is directly proportional to its distance from the neutral axis; hence, if c be a constant quantity, whose value is or is not known, we shall have $\lambda = cy$*

$$\therefore p = \frac{cE_t ky}{l}$$

Or, if we adopt the same notation as in the preceding case, we shall have for the total force tending to produce extension,

$$\Sigma p = \frac{cE_t}{l}(k_1 y_1 + k_2 y_2 + k_3 y_3 + \&c.) \ldots\ldots\ldots (37)$$

Similarly for compression

$$\Sigma p = \frac{cE_c}{l}(k'y' + k''y'' + k'''y''' + \&c.) \ldots\ldots (38)$$

Placing these equal to each other and we have,

$$E_t (k_1 y_1 + k_2 y_2 + k_3 y_3 + \&c.) = E_c (k'y' + k''y'' + k'''y''' + \&c.)$$

or, in the language of the integral calculus,

$$E_t \, \Sigma_0^{y} ydydx = E_c \, \Sigma_{-y}^{0} \, ydydx, \ldots\ldots\ldots\ldots (39)$$

* Comparing this equation with equation (45) gives $c = \frac{dx}{\rho}$

in which y is an ordinate and x an abscissa. Equation (39) enables us to find the position when the form of section is known. In most cases, however, the reduction is not easily made.

Example.—Suppose the sections are rectangular.

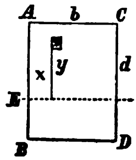


FIG. 28a.

Let $b = AC$,
$d = AB$,
$\frac{E_t}{E_c} = a$, and
$y = AE$ for the superior limit.

Then equation (39) becomes

$$a\int_0^b \int_0^y ydydx = \int_0^b \int_0^{d-y} ydydx, \text{ which reduced becomes}$$

$$ab\frac{y^2}{2} = \frac{b}{2}\left[d-y\right]^2$$

$$\therefore y = \frac{d}{1+\sqrt{a}} \quad \text{.....................(40)}$$

If $a = 1$, $y = \frac{d}{2}$
$a = \infty$, $y = 0$
$a = 0$, $y = d$.

If y is known in equation (40), the ratio of the coefficients of elasticity may easily be found; for, we have from (40)

$$a = \left(\frac{d-y}{y}\right)^2 = \frac{E_t}{E_c} \quad \text{....................(41)}$$

3d. Suppose that the deflecting force is not perpendicular to the axis, and $E_c = E_t = E$.

Let θ = the angle which P makes with the axis of the beam Fig. 29;

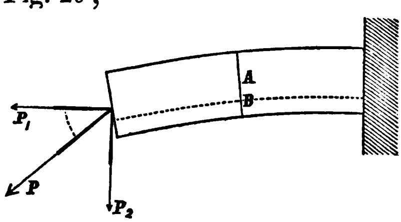


FIG. 29.

$P_1 = P \cos\theta$ = the component of P in the direction of the axis of the beam;

$P_2 = P \sin\theta$ = the component of P perpendicular to the axis of the beam;

h = the distance of the neutral axis from the centre of gravity of the section AB, and K = the transverse section.

The whole force of compression equals the whole force of extension, equations (37) and (38).

$$\therefore P \cos\theta + \frac{cE}{l}\iint_0^v y\,dy\,dx = \frac{cE}{l}\iint_{-v}^0 y\,dy\,dx$$

But the ordinate to the centre of gravity is (see foot-note on page 88),

$$h = \frac{-\iint_0^v y\,dy\,dx + \iint_{-v}^0 y\,dy\,dx.}{K}$$

$$\therefore P \cos\theta = \frac{cE}{l}Kh$$

$$\text{or } h = \frac{Pl}{cEK}\cos\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (42)$$

If $\theta = 90°$, $h = 0$ as before found.

If $\theta = 0$ there is no neutral axis, for the force coincides with the axis of the beam. The equation will show the same result, if the value of $c = \frac{\lambda}{y} = \frac{l}{\rho}$, equation (45), is substituted in the formula, for then ρ would be infinite, for $c = 0$, and h becomes infinite.

4th. Let the law of resistance be according to Barlow's *theory of flexure*, and the deflecting forces normal to the axis of the beam.

Using the same notation as before, also

d_1 = the distance of the most remote fibre from the neutral axis, and

ϕ = the coefficient of longitudinal shearing stress.

Then $\phi\int_0^v x\,dy$ = the resistance to shearing for tension,

and $\phi\int_{-v}^0 x\,dy$ = the resistance to shearing for compression,

and, proceeding as we did to obtain equation (39), we have

$$\frac{T}{d_1}\iint_0^v x\,dy\,dx + \phi\int_0^v x dy = \frac{T}{d_1}\iint_{-v}^0 x\,dy\,dx + \phi\int_{-v}^0 x\,dy. (43)$$

Examples.—Let the sections be rectangular, b = the breadth, d = the depth. Then Eq. (43) becomes

$$\tfrac{1}{2}Td_1 + \phi\, d_1 = \frac{T}{2d_1}(d - d_1)^2 + \phi\,(d - d_1)$$

$$\text{or, } \left(\phi + \frac{Td}{2d_1}\right)(2d_1 - d) = 0;$$

$$\therefore d_1 = \tfrac{1}{2}d \text{ or, } d_1 = -\frac{Td}{2\phi}$$

the former only of which is admissible.

If the value of C were less than that of T, the former would be used instead of the latter in Eq. (43).

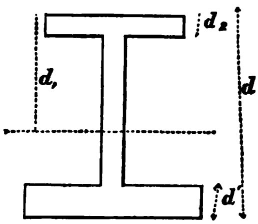


FIG. 30.

If the section is a double **T**, as in Fig. 30, with the notation as in the figure, ϕ will be used in finding the resistance of the vertical rib, and according to Article 75, $\phi\frac{d'}{d-d_1}$ of the lower flange, and $\phi\frac{d_2}{d_1}$ of the upper flange.

It appears from these several cases that the neutral axis passes near the centre of gravity in most practical cases, and it will be assumed that it passes through the centre unless otherwise stated.

CHAPTER IV.

SHEARING STRESS.

89.—GENERAL STATEMENT.—Two kinds of shearing stress are recognized—longitudinal and transverse—both of which have been defined in Article 2. Materials under a variety of circumstances are subjected to this stress—such as, rivets in shears; the rivets in riveted plates; pins and bolts in spliced joints; beams subjected to transverse strains; bars which are twisted; and, in short, all pieces which are subjected to any kind of distorsive stress in which all parts are not equally strained. In the first examples above enumerated, all parts of the section are supposed to be equally strained, but the straining forces act in opposite directions. Shearing may take place in detail, as when plates or bars of iron are cut with a pair of shears, when only a small section is operated upon at a time; or it may be so done as to bring into action the whole section at a time, as in the process of punching holes into metal, where the whole convex surface of the hole is supposed to resist uniformly.

90.—MODULUS OF SHEARING STRENGTH.—The modulus of resistance to shearing is the resistance which the material offers per unit of section to being forced apart when subjected to a shearing stress.

This we call S. The total resistance to ultimate shearing has been found to vary directly as the section; so that if $K =$ the area of the section subjected to this stress the total resistance will be

$$K. S.$$

The value of S has been found for several substances, the principal of which are as follows:—

METALS.

	S in lbs. per square inch.
Fine cast steel *	92,400
Rivet steel †	64,000
Wrought iron *	50,000
Wrought-iron plates punched ‡	51,000 to 61,000
Wrought iron hammered scrap punched §	44,000 to 52,000
Cast iron	30,000 to 40,000
Copper ‖	33,000

WOOD.

With the fibres.

White pine	480
Spruce	470
Fir ¶	592
Hemlock **	540
Oak	780
Locust	1,200

Across the fibres.

Red pine	500 to 800
Spruce	600
Larch ††	970 to 1,700
Treenails, English oak ‡‡	3,000 to 5,000

It will be seen from these results that the shearing strength of wrought iron is about the same as its tenacity; of cast steel it is a little less than its tenacity; of cast iron it is double its tenacity, and about $\frac{2}{5}$ its crushing resistance; and of copper it is about $\frac{3}{4}$ its tenacity.

The following table, which gives the results of some experiments upon punching plate iron, also illustrates the law of resistance, and gives the value of S for that material.

* *Weisbach Mech. and Eng.*, vol. i., p. 407.
† Kirkaldy's *Exp. Inq.*, p. 71.
‡ *Proc. Inst. Mech. Eng.* England, 1858, p. 76.
§ *Proc. Inst. Mech. Eng.* England, 1858, p. 73.
‖ Stoney *on Strains*, vol. ii., p. 284.
¶ Barlow *on the Strength of Materials*, p. 24.
** *Engineering Statics*, Gillespie, p. 33.
†† *Tredgold's Carpentry*, p. 42.
‡‡ Murray *on Shipbuilding Wood and Iron*, p. 94.

TABLE

*Of Experiments on Pumching Plate Iron.**

Diameter of the hole.	Thickness of the plate.	Sectional area cut through.	Total perssure on the punch.	Pressure per square inch of area.
Inch.	Inch.	Square inch.	Tons.	Tons. (Gross.)
0.259	0.437	0.344	8.384	24.4
0.500	0.625	0.982	26.678	27.2
0.750	0.625	1.472	34.768	23.6
0.875	0.875	2.405	55.500	23.1
1.000	1.000	3.142	77-170	24.6

These results give for the value of S from 51,000 lbs. to 61,000 lbs. The total resistance varies nearly as the cylindrical surface of the hole.

APPLICATIONS.

91.—PROBLEM OF A TIE-BEAM.—*To find the relation between the distance AB, Fig. 31, and the depth of a rectangular beam below the notch, so that the total shearing strength shall equal the total tenacity,*

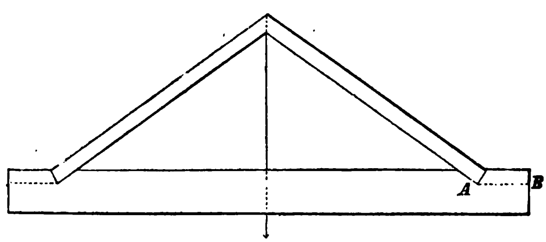


FIG. 31.

Let $h = AB =$ the distance of the bottom of the notch from the end,
$d =$ the remaining depth of the beam,
$k =$ the section of AB,
$K =$ the section below A,
$T =$ the modulus of tenacity, and
$S =$ the modulus of shearing strength:

* *Proceedings Inst. Mech. Eng.*, 1858, p. 76.

Then the condition requires that

$$TK = S\,k,\ \text{but}\ k : K :: h : d$$

$$\therefore \frac{k}{K} = \frac{h}{d} = \frac{T}{S}$$

$$\therefore h = \frac{Td}{S}$$

Example.—For Oak $\frac{T}{S} = \frac{12000}{780} = 15\frac{1}{4}$ nearly; hence AB should be about $15\frac{1}{4}$ times the remaining depth.

92.—RIVETED PLATES.—*Given the diameter of the rivets; it is required to find the distance between them from centre to centre, so that the strength of the rivets for a single row shall equal the strength of the remaining iron in the plates.*

Let d = the diameter of the rivets,
c = the distance between them from centre to centre,
k = the section of the rivet,
K = the remaining section of the plate, and
t = the thickness of the plate.

For iron $T = S$; hence, proceeding as above, we have

$$\frac{k}{K} = \frac{\frac{1}{4}\pi d^2}{t(c-d)} = 1 \therefore c = \frac{0.7854d^2}{t} + d.$$

Examples.—If $t = \frac{1}{2}$ inch, and $d = \frac{1}{2}$ inch;
then $c = 1.2854$, inch,
and $\frac{c-d}{c} = 0.61$.

If $t = \frac{1}{2}$ inch, and $d = \frac{3}{8}$ inch; then $c = 0.8168$ and $\frac{c-d}{c} = 0.541$, which is nearly the value given by Fairbairn for the strength of single riveted plates. See Article 27. To insure this strength the rivet should fit tightly in the hole.

93.—TRANSVERSE SHEARING IN BENT BEAMS.—In Figs. 24 and 25 we considered only the elongations and compressions of the fibres, but in transmitting the strains from the middle to the supports there may be a vertical force at every vertical section,

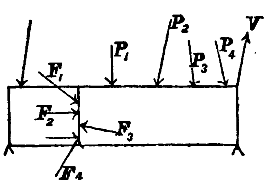


FIG. 32 *a*.

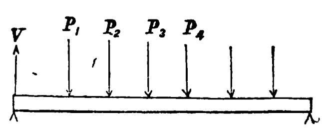


FIG. 32 *b*.

as is indicated by Fig. 26, and which may be shown in a general way from the equations of statics.

In order to simplify the problem, suppose that all the bending forces are in a plane, and let

P, P_1, P_2, &c., be the bending forces,

F, F_1, F_2, &c., be the internal forces in any section of a beam, each of which is the resultant of all the forces concurring at that point.

α, α_1, α_2, &c., the angles which P, P_1, &c., make with the axis of x,

a, a_1, a_2, &c., the angles which F, F_1, F_2, &c., make with the axis of x, and y an axis perpendicular to x.

Then the principles of Statics give the following equations:

$$\left.\begin{aligned} &\Sigma P \cos \alpha + \Sigma F \cos a = 0 \\ &\Sigma P \sin \alpha + \Sigma F \sin a = 0 \\ &\Sigma(Py \cos \alpha - Px \sin \alpha) + \Sigma(Fy \cos a - Fx \sin a) = 0. \end{aligned}\right\} \quad (44)$$

Let x coincide with the axis of the beam, and let all the forces be vertical; or $\alpha = 90°$ or $270°$; then

$$\left.\begin{aligned} &(1) \ldots\ldots\ldots\ldots \Sigma F \cos a = 0 \\ &(2) \ldots\ldots\ldots\ldots \Sigma \pm P + \Sigma F \sin a = 0 \\ &(3) \ldots\ldots\ldots\ldots \Sigma \pm Px + \Sigma Fy \cos a - \Sigma Fx \sin a = 0 \end{aligned}\right\} \quad (44a)$$

The first of these equations shows that the sum of the resisting forces parallel to the axis is zero; or that the total compression equals the total tension. This is equation (35) in another form. The second shows that the sum of the bending forces equals the sum of the vertical components of the resisting forces. If we let S_s represent the total strain, this equation becomes $\Sigma P = \Sigma F \sin a = S_s$, which is the result sought.

That is, *when the bending forces of a beam act vertically to the axis of the beam, the algebraic sum of all the bending forces between one end and the section considered equals the vertical shearing stress in that section.*

The following are some of the more simple cases:

1. Beam fixed at one end and loaded with a weight P at the free end, Fig. 36 .... $S_s = P$

2. Beam fixed at one end and loaded uniformly, Fig. 38, (load being w per unit of length).................. $S_s = wx$

3. Beam supported at its ends and loaded with a weight P at the middle, Fig. 40............................ $S_s = \frac{1}{2} P$

4. Beam supported at its ends and uniformly loaded, Fig. 42, $S_s = \frac{1}{2}wl - wx$.

5. If a beam is supported at its ends, and loaded with several weights P_1, P_2, P_3, etc., as in Fig. 32 *b*, we have for the *shearing stress*.

between the end and $P_1 = V$;
between P_1 and $P_2 \quad = V - P_1$;
between P_2 and $P_3 \quad = V - P_1 - P_2$;
between P_3 and $P_4 \quad = V - P_1 - P_2 - P_3$; etc.

If the weights are equal to each other $= P$, we have $P = P_1 = P_2 = P_3$, etc.; and if there are n of them, and they are symmetrically placed in reference to the centre of the beam, we have

$$V = \tfrac{1}{2}nP.$$

If n is even, we have, at the centre of the beam, the transverse *shearing stress* $= \frac{1}{2}nP - \frac{1}{2}nP = 0$;
and if n is odd, there will be a weight at the centre, and each side of the central weight we have
transverse *shearing stress* $= \frac{1}{2}nP - \frac{1}{2}(n \pm 1)P = \pm \frac{1}{2}P$.

These values are evidently independent of the form or magnitude of the beam. The consideration of the latter enters when we wish to proportion the beam to resist the former.

The development of the third equation gives

$$P_1 x_1 + P_2 x_2 + \&c., + F_1 y_1 \cos a_1 + F_2 y_2 \cos a_2 + \&c.$$
$$-F_1 x' \sin a_1 - F_2 x'' \sin a_2 + \&c. \ldots\ldots\ldots\ldots = 0.$$

Since x_1, x_2, &c., x', x'', &c., are linear quantities, the differential of x_1 equals the differential of x_2; hence we have

$$dx_1 = dx_2 = dx_3 = \&c. \ldots = dx' = dx'' = \&c.$$

Similarly

$$dy_1 = dy_2 = \&c.$$

Hence, by differentiating the above equation, we have

$$\Sigma P\, dx - \Sigma F \sin a\, dx + \Sigma F \cos a\, dy = 0.$$

$$\text{or } \frac{dy}{dx} \Sigma F \cos a = \Sigma F \sin a - \Sigma P.$$

But the first of Eqs. (44*a*) reduces this equation to zero.

$$\therefore \Sigma P = \Sigma F \sin a = S_r.$$

That is, *the vertical shearing stress in a beam when the applied forces are vertical is equal to the first differential coefficient of the moment of the applied forces.*

For example, when a beam is fixed at both ends and loaded uniformly, the moment of the applied forces is

$$\tfrac{1}{8}\, wx\,(4x - 3l)$$

as given in Eq. (112). Hence, according to the above rule we have

$$S_s = \tfrac{1}{2}\, wl - wx.$$

When the bending moment has an algebraic maximum, the moment is greatest where the shearing stress is zero; for the first differential coefficient of the moment of applied forces is the value of the shearing stress, and this placed equal to zero and solved for x will give the point of greatest stress.

The sum of the moments may be represented by a resultant moment.

Let $P'x' = \Sigma Px$; $x'' \Sigma F \sin a = \Sigma Fx \sin a$; and $F'y' = \Sigma Fx \cos a$. Then the second of Eqs. (42a) becomes

$$P'x' - x'' \Sigma F \sin a = F'y'$$

$$\text{or, } P'x' - x'' S_s = F'y'$$

$$\text{or, } P'(x' - x'') = F'y'$$

hence the shearing stress forms a couple with the applied force, or resultant of applied forces.

If the origin of moments be taken in the section considered, x'' will be zero, and we have

$$P'x' = F'y'.$$

or more generally

$$\Sigma Px = \Sigma Py$$

which is the fundamental equation for flexure and rupture of beams under transverse strains.

94.—BENDING DUE TO TRANSVERSE SHEARING.—In order to determine the amount of deflection due to the loading and transverse elasticity, it is necessary to know the law of the distribution of the shearing strain over the cross section. When the body is sheared off without deflection, as in the case of rivets, and other cases where the shearing force acts on the plane of resistance, the stress is uniformly distributed over the cross section; but this is not necessarily the case when the shearing stress is accompanied by flexure.

It will be shown in Chapter IX. that the shearing stress is zero at the upper and lower surfaces, and increases from these points towards the neutral axis, at which point it is a maximum. It will be found in rectangular beams that the decrease of the shearing stress from the neutral axis varies as the square of the ordinate—Equation (210)—and hence the shearing stress may be represented by the area of a common parabola, the diminution being represented by the external part of the same parabola.

Hence, if

$b =$ the breadth of a beam,
$d =$ the depth,
$A =$ the area of a section $= bd$,
$l =$ the length,
$E_s =$ the coefficient of transverse elasticity,
$P =$ the applied weight, and
$\Delta_s =$ the deflection; we have

$\frac{2}{3}E_s bd =$ the total resistance to transverse shearing.

The deflection will evidently increase directly as the length; hence, if the beam be fixed at one end and the weight be applied at the free end

$$\Delta_s = \frac{3}{2}\frac{Pl}{bdE_s}$$

If the beam be supported at its ends and loaded at the middle, we have

$$\Delta_s = \frac{\frac{1}{2}P.\frac{1}{2}l}{\frac{2}{3}b\,d\,E_s} = \frac{3}{8}\frac{P\,l}{AE_s}$$

If the beam be supported at its ends and uniformly loaded, we have

$$\Delta_s = \frac{\int_0^{\frac{1}{2}l}(\frac{1}{2}\,wl - wx)\,dx}{\frac{2}{3}A\;E_s} = \frac{3wl^2}{16A\;E_s} = \frac{3\,Wl}{16A\;E_s}$$

The total deflection depends upon the elongation and compression of the elements, as well as upon transverse shearing, and hence involves both E and E_s. By comparing the values of the deflections above given, with those of the corresponding cases in the next chapter, it will be seen that the deflection due to transverse shearing has but little relative effect for long beams, but for very short ones it becomes the more important element.

Example.—Required the deflection of a rectangular beam due to transverse shearing which is supported at its ends and loaded uniformly over its whole length, when $b = 4$ inches, $d = 10$ inches, $l = 8$ feet; w (the load per foot of length) $= 500$ pounds, and $E_s = 45,000$ pounds.

Weisbach says: "The coefficient of transverse elasticity is assumed to be $\frac{1}{3}$ E" (Weisbach, *Mechanics of Engineering*, Vol. I., p. 522). This is nearly $\frac{2}{5}$ E, the value found theoretically for amorphous bodies, but for fibrous bodies, such as wood, the transverse elasticity is not the same in the different directions of the layers, so that it has not a specific value.

There is a longitudinal shearing at every point of a beam where there is transverse shearing, but the deflection which arises from it is small. The analysis of these cases is reserved for Chapter IX., since a portion of it depends upon the analysis for flexure.

95. Shearing resistance to torsion.—When a solid is twisted the consecutive transverse sections of elements slip over each other, and for small angles of torsion, such as are only admitted in practice, the law of strains is comparatively simple, as is shown from theoretical considerations, and which is confirmed by experiment. This law is: the strains increase directly as the distance from the axis of the piece, as stated in Chapter VIII., and is applicable to wood and other fibrous and granular solids.

But when the elastic limit is passed the case becomes very complex. All the elements which originally were rectilinear become helices, except those at the axis. The outer elements thus become elongated, and by their elastic resistance produce compression upon those near the axis. There will also be a lateral contraction of those elements which are elongated. The transverse sections which were originally plane will become warped. As the strain is increased the outer elements actually slip over each other, and thus lose, in a great degree, their power of resistance, and throw greater strains upon those nearer the axis, until finally the elements are sheared apart. During this process, shearing strains may exist in any direction—longitudinally, lateral, and tangential.

In the more ductile metals rupture may take place slowly, and the final fracture be nearly a plane which will be perpendicular to the axis of the piece; but in brittle metals, such as highly tempered steel and most qualities of unmalleableized cast-iron, rupture takes place suddenly with a "snap" when under strain, and with only a small amount of torsion. In such cases the fracture is irregular and oblique to the axis. There is little or no appearance of shearing, for rupture takes place with only a small amount of shearing.

The conditions of ultimate rupture do not appear to be governed by definite mathematical laws; and hence it might appear useless to subject them to *hypothetical* laws; but the laws which are assumed are sufficiently exact for practical cases when the material is not overstrained.

Remark.—It is fortunate that for practical purposes it is not necessary to know the *exact* condition of the strains within a piece which is used in a structure, for it is impossible to construct an equation which will represent every possible case with mathematical exactness. Bodies are infinitely diversified. Some may be subjected to internal strains from the process of manufacture. These may be caused by forging some parts more than others; but especially by unequal cooling. The effect of an external load may be to increase the intensity of some of these strains and relieve others. We also see that a simple stress may produce various strains: and hence when the bodies are free from internal strains, and are perfectly homogeneous, the analysis which considers all the changes becomes exceedingly refined.

We know by long experience that it is only necessary to keep within certain limits, and these limits can easily be determined.

In the analysis of the more simple cases we consider only one distortion at a time. Thus, in stretching a piece, we consider the more apparent phenomenon—that of elongation—but at the same time there is a lateral contraction which, in practice, is so small that we disregard it, but which in a thorough analysis must be considered. Also in regard to flexure, we usually consider only the effect of the elongation of some of the fibres and the compression of others, as in the following chapter, but this change is necessarily accompanied by others, which in ordinary cases may be disregarded. The same remarks apply to torsion and to transverse shearing.

The analysis which determines the relation between strains and stresses in elastic bodies has given rise to a department of mathematical physics called the Mathematical Theory of Elasticity, which has been developed by M. Lamé (*Leçons de la Théorie Mathématique de l'Elasticité des Corps, Solides*, Paris, 1852); M. Louiville (*Journal Louiville*, 1863, etc.); M. Kirchoff (*Ueber das Gleichgewicht und Bewegung einer unendlich dunnen elastiches Stabes*,* Journal de Crelle, tome 56, p. 285); M. Maxwell (*On the Equilibrium of Elastic Bodies;* Transactions of the Royal Society of Edinburgh, vol. xx., 1853, p. 87, etc.); M. Cauchy (*Exercises d'analysis et de Physique Mathématique; Comptes Rendus*, etc.), and others.

* On the equilibrium and movement of an infinitely slender elastic rod.

CHAPTER V.

FLEXURE.

96. ELASTIC CURVE.

WHEN a beam is bent by a transverse strain, equilibrium is established between the external and internal forces; or, to be more specific, all the external forces to the right or left of any transverse section are held in equilibrium by the elastic resistances of the material in the section. When in this state the curve assumed by the neutral axis is called the *elastic curve*.

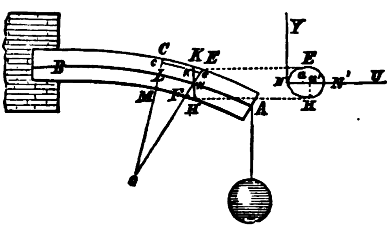


FIG. 33.

We will first find the equation of the elastic curve according to the conditions of the "Common Theory," following substantially the method originally given by Navier. Let Fig. 33 represent a beam, fixed at one end, or supported in any manner, and deflected by a weight, P, or by any number of forces. AB is the neutral axis. Take the origin of co-ordinates at B (or at any other point on the neutral axis), and let x be horizontal and coincide with the axis of the beam before flexure, y vertical and u perpendicular to the plane of xy. The transverse sections CM and EF being consecutive and parallel before flexure, will meet after flexure, if sufficiently prolonged in some point, as o. Through N draw KH parallel to CM; then will ke be the elongation of a fibre whose original length was ck. We have the following notation:—

$dx = LN =$ the distance between consecutive sections,

$y' = Ne =$ any ordinate of the surface,

$u = Na$ or Na'.

$b = NN' =$ the limiting value of u,

$f(y', u) =$ equation of the transverse section,

$dy'du$ = the transverse section of a fibre,

$f(x, y)$ = the equation of the neutral axis,

$\rho = ON$ = the radius of curvature at N,

p = the force necessary to elongate any fibre an amount equal to λ when applied in the direction of its length,

$\lambda = ke$,

I = the moment of inertia of the section,

E = the coefficient of elasticity of the material, which is supposed to be the same for extension and compression,

ΣPx = a general expression for the moment of applied forces.

We suppose that the strain is within the elastic limit, and establish the algebraic equation on the condition that the sum of the moments of the applied or deflecting forces equals the sum of the moments of the resisting forces. We also assume that the neutral axis coincides with the centre of the transverse sections of the beam.

By the similarity of the triangles LON and kNe, we have

$$ON : Ne :: LN : ke, \text{ or } \rho : y' :: dx : \lambda$$

$$\therefore \lambda = \frac{y'}{\rho} dx \quad \ldots\ldots\ldots\ldots (45)$$

The force necessary to produce this elongation is (see Equation (3)),

$$p = E\, dy'du\, \frac{\lambda}{dx};$$

which becomes, by substituting λ from (45),

$$p = \frac{E}{\rho} y'\, dy'\, du \quad \ldots\ldots\ldots\ldots (46)$$

and the moment of this force is found by multiplying it by y';

$$\therefore py' = \frac{E}{\rho} y'^2\, dy'du \quad \ldots\ldots\ldots\ldots (47)$$

The total moment of all the resisting forces to extension and compression is found by integrating Equation (47) so as to include the whole transverse section, and this will equal the sum of the moments of the applied forces:

$$\therefore \frac{E}{\rho}\left[\int\int_0^{+y'} y'^2 dy'du + \int\int_{-y'}^{0} y'^2 dy'du\right] = \Sigma Px$$

$$\text{or } \frac{E}{\rho}\int\int_{-y'}^{+y'} y'^2 dy' du = \Sigma Px \ldots\ldots\ldots\ldots\ldots\ldots (48)$$

The quantity $E\int\int y'^2 dy' du$, which depends upon the form of the transverse section and nature of the material, is called the *moment of flexure.*

The quantity $\int\int y'^2 dy' du$, when taken between limits so as to include the whole transverse section, is called the moment of inertia of the surface.* Calling this I and Equation (48) becomes

$$\frac{EI}{\rho} = \Sigma Px \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (49)$$

which is the equation of the *elastic curve.*

An exact solution of Equation (49) is not easily obtained in practice, except in a few very simple cases; but when the deflection is small an approximate solution, which is generally comparatively simple and always sufficiently exact, is easily found.

$$\text{We have, } \rho = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{d^2y\, dx} = \frac{dx^2\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{d^2y}$$

$$= \frac{dx^2}{d^2y} \text{ nearly, since for small deflections}$$

$\frac{dy}{dx}$ (which is the tangent of the angle which the tangent line to the curve makes with the axis of x) is small compared with unity, and hence may be omitted. Hence equation (49) becomes

$$EI\frac{d^2y}{dx^2} = \Sigma Px \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (50)$$

which is the general *approximate* Equation of the neutral axis.

97. THE MOMENT OF INERTIA of all transverse sections of a prismatic beam is constant, and hence I is constant for prismatic beams.

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* See Appendix.

For a rectangle, as Fig. 34, we have

$$I = \int_0^b \int_{-\frac{1}{2}d}^{+\frac{1}{2}d} y^2 dy du = \tfrac{1}{12} bd^3 \ldots\ldots\ldots (51)$$

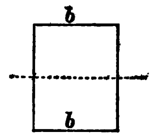

FIG. 34.

For a circle, the origin of coördinates being at the centre;

$$y = \rho \sin \theta$$
$$dy\, du = \rho\, d\rho\, d\theta$$

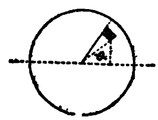
FIG. 35.

$$\therefore I = \int_0^r \int_0^{2\pi} \rho^3 d\rho\, d\theta \sin^2\theta = \tfrac{1}{4}\pi r^4 \ldots\ldots (52)$$

SPECIAL CASES OF PRISMATIC BEAMS.

98. REQUIRED THE EQUATION OF THE NEUTRAL AXIS, AMOUNT OF DEFLECTION, AND SLOPE OF THE CURVE OF A PRISMATIC BEAM, WHEN SLIGHTLY DEFLECTED, AND SUBJECTED TO CERTAIN CONDITIONS AS FOLLOWS:

99. CASE I.—SUPPOSE A HORIZONTAL BEAM IS FIXED AT ONE EXTREMITY AND A WEIGHT P RESTS UPON THE FREE EXTREMITY; REQUIRED THE EQUATION OF THE NEUTRAL AXIS AND THE TOTAL DEFLECTION.

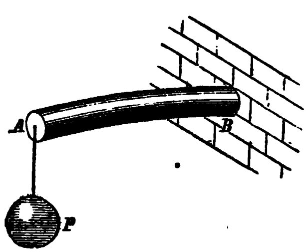

FIG. 36.

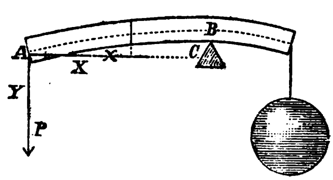

FIG. 37.

The beam may be fixed by being embedded firmly in a wall, as in Fig. 36, or by resting on a fulcrum and having a weight applied on the extended part, which is just sufficient to make the curve horizontal over the support, as in Fig. 37. The latter case more nearly realizes the mathematical condition of fixedness. In either case let

$l = AB =$ the length of the part considered,
$i =$ the inclination of the curve at any point, and
$\Delta = BC =$ the total deflection.

Take the origin of coördinates at the free end, A; x horizontal, y vertical and positive downwards. The moment of P on any section distant x from A is Px, which is the second member of Equation (50) in this case. Hence Eq. (50) becomes

$$EI\frac{d^2y}{dx^2} = Px \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (53)$$

Multiply both members by dx and integrate, and we have

$$EI\frac{dy}{dx} = \tfrac{1}{2}Px^2 + C_1 \ldots\ldots\ldots\ldots\ldots\ldots (54)$$

When the deflections are small, the length of the beam remains sensibly constant, hence for the point B, $x = l$; and at the fixed end $\frac{dy}{dx} = 0$. Substitute these values in Eq. (54), and we find $C_1 = -\tfrac{1}{2}Pl^2$, and (54) gives

$$\frac{dy}{dx} = \frac{P}{2EI}(x^2 - l^2) = \text{tang } i \ldots\ldots\ldots\ldots\ldots (55)$$

The integral of Equation (55) is

$$y = \frac{P}{6EI}(x^3 - 3l^2x) + C_2$$

But the problem gives $y = 0$ for $x = 0 \therefore C_2 = 0$;

$$\therefore y = \frac{P}{6EI}(x^3 - 3l^2x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56)$$

which is the equation of the neutral axis, according to the common theory, and may be discussed like any other algebraic curve.

The greatest slope is at A, to find which make $x = 0$ in Equation (55)

$$\therefore \text{tang } i \text{ (at the free end)} = -\frac{Pl^2}{2EI}$$

The greatest distance between the curve and the axis of x is at B, to find which make $x = l$ in Equation (56), and we have

$$y = \Delta = -\frac{Pl^3}{3EI} \quad \text{.........................} (57)$$

In this case we have

$$Ss = EI\frac{d^3y}{dx^3} = \frac{d(Px)}{dx} = P \quad \text{.................} (57a)$$

That is, the transverse shearing strain is uniform over the whole length and equal to the load at the free end.

Differentiating again gives

$$\frac{d^4y}{dx^4} = 0;$$

that is, the increment of transverse shearing is zero.

If y were positive upward, everything else remaining the same, the second member of Eq. (53) would have been negative, for it is a principle in the Differential Calculus that when the curve is concave to the axis of x, the second differential coefficient and the ordinate must have contrary signs. This would make tang i and Δ positive. It will be a good exercise for the student to solve this and other problems by taking the origin of coördinates at different points, only keeping x horizontal and y vertical. For instance, take the origin at B; at C; at the point where the free end of the beam was before deflection; at the middle of the beam; or at any other point.

***Example.*—If $l = 5$ ft., $b = 3$ in., $d = 8$ in., $E = 1{,}600{,}000$ lbs., and $P = 5{,}000$ lbs.; required the slope at the free end and at the middle, and the maximum deflection.**

100. CASE II.—SUPPOSE THAT THE BEAM IS FIXED AT ONE END, IS FREE AT THE OTHER, AND HAS A LOAD UNIFORMLY DISTRIBUTED OVER ITS WHOLE LENGTH.—The beam may be fixed as before, as shown in Figs. 38 and 39.

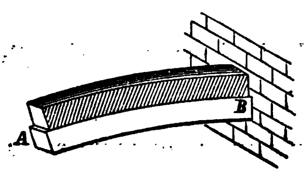

FIG. 38.

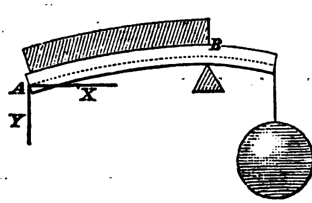

FIG. 39.

Let w = the load on a unit of length. This load may be the weight of the beam, or it may be an additional load.

$W = wl$ = the total load.

Take the origin at A.

Then wx = the load on a distance x, and

$\frac{1}{2}wx^2$ = the moment of this load on a section distant x from A.

Hence Equation (50) becomes

$$EI\frac{d^2y}{dx^2} = \tfrac{1}{2}wx^2 \qquad (58)$$

$$\therefore \frac{dy}{dx} = \frac{w}{6EI}(x^3 - l^3) = \text{tang}\, i \qquad (59)$$

$$\therefore y = \frac{w}{24EI}(x^4 - 4l^3x) \qquad (60)$$

$$\text{and } \Delta = -\frac{wl^4}{8EI} = -\frac{Wl^3}{8EI} \qquad (61)$$

In which $\frac{dy}{dx} = 0$ for $x = l \therefore C_1 = -\frac{wl^3}{6EI}$,

$y = 0$ for $x = 0 \therefore C_2 = 0$, and

$y = \Delta$ for $x = l$.

If the origin of coördinates were at the fixed end, ΣPx in the first case would be $P(l - x)$, and in the second $\frac{w}{2}(l - x)^2$. The student may reduce these cases and find the constants of integration. This case may be further modified for practice by taking the origin of coördinates at different points.

From Eq. (58) we have

$$Ss = EI\frac{d^3y}{dx^3} = wx \qquad (62a)$$

Also

$$d\,Ss = EI\frac{d^4y}{dx^3} = w\,dx;$$

that is, the increment of shearing is the load per unit of length multiplied by the increment of length.

101. CASE III.—LET THE BEAM BE FIXED AT ONE END AND A LOAD UNIFORMLY DISTRIBUTED OVER ITS WHOLE LENGTH, AND A

WEIGHT ALSO APPLIED AT THE FREE END.—This is a combination of the two preceding cases, and is represented by Figs. 36 and 37, in which the weight of the beam is the uniform load.

$$\therefore EI\frac{d^2y}{dx^2} = Px + \tfrac{1}{2}wx^2;$$

$$\text{and } \Delta = -\frac{l^3}{3EI}(P + \tfrac{3}{8}W) \text{.....................(62)}$$

hence the deflection of a beam fixed at one end and free at the other, and uniformly loaded, is $\frac{3}{8}$ as much as for the same weight applied at the free end.

102. CASE IV.—LET THE BEAM BE SUPPORTED AT ITS ENDS AND A WEIGHT APPLIED AT ANY POINT.—Figs. 40 and 41 represent the case.

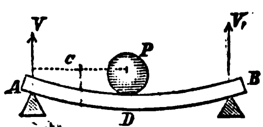

FIG. 40.

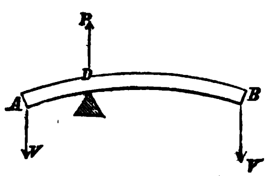

FIG. 41.

Let the reaction of the supports be V and V_1. Take the origin at A over the support, and let $AD = c =$ the abscissa of the point of application of P.

$$\text{Then, } V = \frac{l-c}{l}P, \text{ and } V_1 = \frac{c}{l}P.$$

The case is the same as if a beam rested on a support at D, and weights equal to V and V_1 were suspended at the ends.

For the part AD, Equation (50) becomes:—

$$EI\frac{d^2y}{dx^2} = -Vx = -\frac{l-c}{l}Px \text{..............(63)}$$

$$\therefore \frac{dy}{dx} = -\frac{P(l-c)}{2lEI}x^2 + C_1 \text{....................(64)}$$

$$\text{and } y = -\frac{P(l-c)}{6lEI}x^3 + C_1x + (C_2 = 0) \text{...........(65)}$$

in the last of which, $y = 0$ for $x = 0$ $\therefore$ $C_2 = 0$ as indicated.

For the part DB, the origin of coördinates remaining at A, we have:—

$$EI\frac{d^2y}{dx^2} = -Vx + P(x-c) = Pc\frac{x-l}{l} = -V_1(l-x) \ldots (66)$$

$$\therefore \frac{dy}{dx} = \frac{Pc}{2lEI}(x^2-2lx) + C' \ldots\ldots (67)$$

$$\text{and } y = \frac{Pc}{6lEI}(x^3-3lx^2) + C'x + C'' \ldots\ldots (68)$$

To find the constants, make $x = c$ in equations (64) and (67) and place them equal to each other; do the same with (65) and (68); and also observe that in (68) $y = 0$ for $x = l$. These conditions establish the three following equations:—

$$-\frac{Pc^2(l-c)}{2lEI} + C_1 = \frac{Pc^2}{2lEI}(c-2l) + C'$$

$$-\frac{Pc^3(l-c)}{6lEI} + C_1c = \frac{Pc^3}{6lEI}(c-3l) + C'c + C''$$

$$0 = -\frac{Pcl^2}{3EI} + C'l + C''$$

From these we find

$$C_1 = \frac{Pc}{6EIl}(c^2 + 2l^2 - 3cl)$$

$$C' = \frac{Pc}{6EIl}(c^2 + 2l^2)$$

$$C'' = -\frac{Pc^3}{6EI}$$

Hence, for the part AD we have

$$EI\frac{d^2y}{dx^2} = -\frac{l-c}{l}Px,$$

$$\frac{dy}{dx} = -\frac{P(l-c)}{2lEI}x^2 + \frac{Pc}{6EIl}(c^2 + 2l^2 - 3cl).$$

$$\text{or, } \frac{dy}{dx} = \frac{P}{6EIl}\left[(-3l+3c)x^2 + c^3 + 2cl^2 - 3c^2l\right] \ldots\ldots (69)$$

$$y = \frac{P}{6EIl}\left[(c-l)x^3 + (c^3+2l^2-3cl)cx\right] \ldots\ldots (70)$$

To find the maximum deflection, if c is greater than $\frac{1}{2}l$, make $\frac{dy}{dx} = 0$ in (69) and find x; then substitute the value thus found in Equation (70). If $c < \frac{1}{2}l$ make $\frac{dy}{dx} = 0$ in Equation (67) and substitute the value thus found in Equation (68).

If D is at the middle of the length, make $c = \frac{1}{2}l$ in equations (63), (69), and (70); and we have for the curve AD

$$EI\frac{d^2y}{dx^2} = -\tfrac{1}{2}Px \qquad (71)$$

$$\frac{dy}{dx} = \frac{P}{16EI}(l^2-4x^2),$$

$$y = \frac{P}{48EI}(3l^2x-4x^3) \qquad (72)$$

$$\text{and } \varDelta = \frac{Pl^3}{48EI}(\text{if } x = \tfrac{1}{2}l \text{ in } (72)) \qquad (73)$$

The greatest stress is at the centre, and the maximum moment is found by making $x = \frac{1}{2}l$ in the second member of Equation (71). Hence, *the maximum moment is*

$$\tfrac{1}{4}Pl \qquad (73a)$$

In this case the curve DB is of the same form as AD, but its equation will not be the same unless the origin of coördinates be taken at the other extremity of the beam.

From Eq. (71) we have

$$Ss = EI\frac{d^3y}{dx^3} = -\tfrac{1}{2}P.$$

Also $d\,Ss = 0$.

103. CASE V.—SUPPOSE THAT A BEAM IS SUPPORTED AT OR NEAR ITS EXTREMITIES, AND THAT A LOAD IS UNIFORMLY DISTRIBUTED OVER ITS WHOLE LENGTH.

No account is made of the small portion of the beam (if any) which projects beyond the supports. The distance between the supports is the length of the beam which is considered.

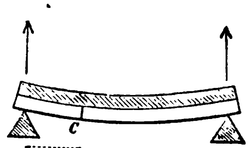

FIG. 42.

Let the notation be the same as in the preceding cases.

Then $V = \frac{1}{2}wl = \frac{1}{2}W =$ the weight sustained by each support;

$Vx = \frac{1}{2}wlx =$ the moment of V on any section, as c;

wx is the load on x, and the lever arm of this load is the horizontal distance from its centre to the section c, or $\frac{1}{2}x$; hence its moment is $\frac{1}{2}wx^2$, and the total moment is the difference of the two moments. Hence Equation (50) becomes

$$EI\frac{d^2y}{dx^2} = \frac{1}{2}w(-lx + x^2) \dots\dots\dots\dots (74)$$

$$\therefore \frac{dy}{dx} = \frac{w}{24EI}(-6lx^2 + 4x^3 + l^3);$$

$$y = \frac{w}{24EI}(-2lx^3 + x^4 + l^3x) \dots\dots\dots (75)$$

and if $x = \frac{1}{2}l$ in (75), $y = \Delta = \dfrac{5wl^4}{384EI} = \dfrac{5\,Wl^3}{384EI} \dots\dots (76)$

In these equations $\dfrac{dy}{dx} = 0$ for $x = \frac{1}{2}l, \therefore C_1 = \dfrac{wl^3}{24EI}$;

and $y = 0$ for $x = 0$, $\therefore C_2 = 0$.

$$Ss = \frac{1}{2}wl - wx.$$

$$dSs = EI\frac{d^4y}{dx^4} = -wdx.$$

104. CASE VI.—LET THE BEAM BE SUPPORTED AT ITS ENDS, UNIFORMLY LOADED, AND ALSO A LOAD MIDWAY BETWEEN THE SUPPORTS.

This case is a combination of the two preceding ones, and may be represented by Fig. 40; for the weight of the beam may be the uniform load. Hence

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}Px + \frac{1}{2}wx^2 - \frac{1}{2}wlx \dots\dots\dots\dots (77)$$

$$\Delta = \frac{l^3}{48EI}\left[P + \frac{5}{8}W\right] \dots\dots\dots\dots\dots\dots (78)$$

Experiments on the deflection of beams are generally made in accordance with this case. If the beam be rectangular, we have from Equation (51),

$$I = \frac{1}{12} bd^3, \text{ which in (78) gives}$$

$$\Delta = \frac{l^3}{4Ebd^3}\left[P + \frac{5}{8}W\right] \ldots\ldots\ldots\ldots\ldots (79)$$

$$\therefore E = \frac{l^3}{4\Delta bd^3}\left[P + \frac{5}{8}W\right] \ldots\ldots\ldots\ldots\ldots (80)$$

According to Equation (79) the deflection of rectangular beams varies as the *cube of the length;* and inversely as the *breadth* and *cube of the depth,* and directly as the weight applied.

In making an experiment to determine E, the beam is weighed, and that portion of it which is between the supports and unbalanced will be W, and all the quantities except E may be directly measured. If E be known, we may measure or assume all but one of the remaining quantities, and solve the equation to find the remaining quantity, as the following examples will illustrate:—

Examples.—1. If a rectangular beam, 5 feet long, 3 inches wide, and 3 inches deep, is deflected $\frac{1}{10}$ of an inch by a weight of 3,000 lbs. applied at the middle; required the coefficient of elasticity. E = 20,000,000 lbs.

2. If $b = 2$ inches, $d = 4$ inches, and $l = 6$ feet, the weight of the beam 144 lbs., and a weight P=10,000 lbs. placed at the middle of the beam deflects it $\frac{1}{2}$ an inch; required E. E = 14,711,220 lbs.

3. A joist, whose length is 16 feet, breadth 2 inches, depth 12 inches, and coefficient of elasticity 1,600,000 lbs., is deflected $\frac{1}{2}$ inch by a weight in the middle; required the weight; the weight of the beam being neglected.
Ans. P = 1,562 lbs.

4. An iron rectangular beam, whose length is 12 feet, breadth $1\frac{1}{2}$ inch, coefficient of elasticity 24,000,000 lbs., has a weight of 10,000 lbs. suspended at the middle; required its depth that the deflection may be $\frac{1}{480}$ of its length.
Ans. 8.8 in.

5. A rectangular wooden beam, 6 inches wide and 30 feet long, is supported at its ends. The coefficient of elasticity is 1,800,000 lbs.; the weight of a cubic foot of the beam is 50 lbs.; required the depth that it may deflect 1 inch from its own weight.

How deep must it be to deflect $\frac{1}{400}$ of its length?

6. A cylindrical beam, whose diameter is 2 inches, length 5 feet, weight of a cubic inch of the material 0.25 lb., is deflected $\frac{3}{8}$ of an inch by a weight P = 3,000 lbs. suspended at the middle of the beam. Required the coefficient of elasticity.

To solve this substitute $I = \frac{1}{4}\pi r^4$ (Equation (52)) in Equation (78). This gives

$$E = \frac{l^3}{12\varDelta\pi r^4}\left[P + \frac{5}{8}W\right]$$

7. Required the depth of a rectangular beam which is supported at its ends, and so loaded at the middle that the elongation of the lowest fibre shall equal $\frac{1}{1400}$ of its original length. (Good iron may safely be elongated this amount.)

Equations (49) and (73*a*) become $\frac{EI}{\rho} = \frac{1}{4}Pl. \therefore \rho = \frac{4EI}{Pl}$. In this substitute the value of I, Equation (51), and it becomes

$$\rho = \frac{Ebd^3}{3Pl}. \quad \text{By the problem find } \rho = 700d$$

$$\therefore d = \sqrt{\frac{2100Pl}{Eb}}.$$

8. Required the radius of curvature at the middle point of a wooden beam, when $P = 3{,}000$ lbs.; $l = 10$ ft.; $b = 4$ in.; $d = 8$ in.; and $E = 1{,}000{,}000$ lbs.

Equations (49) and (73*a*) give $\rho = \frac{EI}{\frac{1}{4}Pl} = \frac{1{,}000{,}000 \times \frac{1}{12} \times 4 \times 8^3}{\frac{1}{4} \times 3{,}000 \times 10 \times 12} = 1{,}896$ inches.

9. Let the beam be iron, supported at its ends. Let $b = 1$ in., $d = 2$ in., $l = 8$ ft., $E = 25{,}000{,}000$ lbs. Required the radius of curvature at the middle when the deflection is $\frac{1}{2}$ of an inch. Use Eqs. (49) and (73) for P at the middle.

$$\therefore \rho = \frac{EI}{\frac{1}{4}Pl} = \frac{E.I}{\frac{1}{4} \times \frac{48E.I.\varDelta.l}{l^3}} = \frac{l^2}{12\varDelta} = 3{,}840 \text{ inches};$$

from which it appears that it is independent of the breadth and depth.

10. The centrifugal force caused by a load moving over a deflected beam may be found from the expression $\frac{mv^2}{\rho}$, in which m is the mass of the moving load, v its velocity in feet per second, and ρ the radius of curvature of the beam. (See Mechanics.)

11. All these problems may be applied to beams fixed at one end, and P applied at the free end, or for a load uniformly distributed over the whole length, by using the equations under Cases I., II., and III.

105. CASE VII.—LET THE BEAM BE FIXED AT ONE EXTREMITY, SUPPORTED AT THE OTHER, AND HAVE A WEIGHT, P, APPLIED AT ANY POINT.

The beam may be fixed by being encased in a wall, Fig. 43, or by extending it over a support and suspending a weight on the extended part sufficient to make the beam horizontal over the support, Fig. 44; or by resting a beam whose length is $2l$

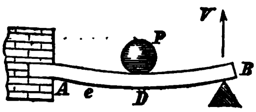

FIG. 43.

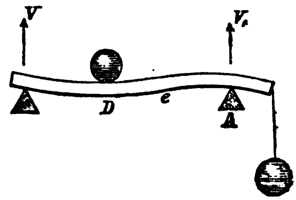

FIG. 44.

on three equidistant supports, and having two weights, each equal to P, resting upon it at equal distances from the central

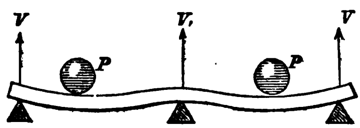

FIG. 45.

support, Fig. 45. In the latter case each half of the beam fulfils the condition of the case.

Let $l = AB$, Fig. 43, be the part considered,

$V =$ the reaction of the support,

$nl = AD =$ the abscissa of P, and

$f =$ the deflection of the beam at D.

Take the origin at A, the fixed end. We may consider that the curve DB is caused by the reaction of V, while all the forces at the left of P hold the beam for V to produce its effect. Similarly the curve AD is produced by the reaction V and the weight P, while all the forces at the left of them hold the beam. In all cases we may consider that the applied forces on one side of the transverse section are in equilibrium with the resisting forces of tension and compression in the section. It is well also to observe that *the origin of moments is at the centre of the transverse section*, while the origin of coördinates may be at any point.

For the curve AD we have, observing that $\frac{dy}{dx} = 0$ for $x = 0$, and $y = 0$ for $x = 0$:—

$$EI\frac{d^2y}{dx^2}=P(nl-x)-V(l-x)\dots\dots\dots\dots\dots(82)$$

$$EI\frac{dy}{dx}=P\left(nlx-\frac{x^2}{2}\right)-V\left(lx-\frac{x^2}{2}\right)\dots\dots\dots\dots(83)$$

$$EIy=P\left(\frac{nlx^2}{2}-\frac{x^3}{6}\right)-V\left(\frac{lx^2}{2}-\frac{x^3}{6}\right)\dots\dots\dots\dots(84)$$

For the point D, we have, by making $x=nl$,

$$\frac{dy}{dx}=\tan i=[\tfrac{1}{2}n^2P-(n-\tfrac{1}{2}n^2)V]\frac{l^2}{EI}\dots\dots\dots(85)$$

$$y=f=[\tfrac{1}{3}n^3P-(\tfrac{1}{2}n^2-\tfrac{1}{6}n^3)V]\frac{l^3}{EI}\dots\dots\dots\dots(86)$$

For the curve DB, observe that $\frac{dy}{dx}=\text{tang}\,i$ for $x=nl$, and $y=f$ for $x=nl$, using for their values (85) and (86) in determining the constants in the following equations, and we have:—

$$EI\frac{d^2y}{dx^2}=-V(l-x)\dots\dots\dots\dots\dots\dots\dots(87)$$

$$EI\frac{dy}{dx}=\tfrac{1}{2}Pn^2l^2-V\left(lx-\frac{x^2}{2}\right)\dots\dots\dots\dots\dots(88)$$

$$EI\,y=(\tfrac{1}{2}x-\tfrac{1}{6}nl)Pn^2l^2-V\left(\frac{lx^2}{2}-\frac{x^3}{6}\right)\dots\dots\dots(89)$$

To find the reaction V, observe that $y=0$, for $x=l$ in (89), and we obtain:—

$$0=(3-n)Pn^2l^3-2Vl^3;$$

$$\therefore V=\tfrac{1}{2}n^2(3-n)P\dots\dots\dots\dots\dots(90)$$

By substituting this value of V in the preceding equations, they become completely determined. For the curve AD we have:—

$$EI\frac{d^2y}{dx^2}=P[nl-x-\tfrac{1}{2}n^2(3-n)(l-x)]\dots\dots\dots(91)$$

$$\frac{dy}{dx}=\frac{P}{4EI}[4nlx-2x^2-n^2(3-n)(2lx-x^2)]\dots\dots(92)$$

$$y = \frac{P}{12EI}[6nlx^2 - 2x^3 - n^2(3-n)(3lx^2 - x^3)] \ldots (93)$$

and for the curve DB:—

$$EI\frac{d^2y}{dx^2} = -\tfrac{1}{2}Pn^2(3-n)(l-x) \ldots\ldots\ldots\ldots (94)$$

$$\frac{dy}{dx} = \frac{Pn^2}{4EI}[2l^2 - (3-n)(2lx - x^2)] \ldots\ldots\ldots\ldots (95)$$

$$y = \frac{Pn^2}{12EI}[(6x - 2nl)l^2 - (3lx^2 - x^3)(3-n)] \ldots (96)$$

The points of greatest strain in these curves are where the sum of the moments of applied forces is greatest, and this is greatest when the second members of (91) and (94) are greatest. Neither of these expressions have an algebraic maximum, and hence *we must find by inspection that value of x which will give the greatest value of the function within the limits of the problem.* Equation (91) has two such values, one for $x = 0$, the other for $x = nl$, and Equation (94) has one such for $x = nl$, which value will reduce (91) and (94) to the same value.

Making $x = 0$ in (91) gives for the moment of maximum strain,

$$\Sigma Px = \tfrac{1}{2}Pl\,[2n - 3n^2 + n^3] \ldots\ldots\ldots\ldots (97)$$

For the moment of strain at P, make $x = nl$, in Eq. (91) or Eq. (94), and we have

$$\Sigma Px = \tfrac{1}{2}\,Pln^2\,[-3 + 4n - n^2] \ldots\ldots\ldots\ldots (98)$$

To find where P must be applied so that the strain at the point of application shall be greater than if applied at any other point, we must find the maximum of (98):—

$$\therefore \frac{d\Sigma Px}{dn} = 0 = -6n + 12n^2 - 4n^3 \ldots\ldots\ldots\ldots (99)$$

$$n = 0.634+ \ldots\ldots\ldots\ldots\ldots\ldots (100)$$

or the force must be applied at more than $\frac{63}{100}$ of the length of the beam from the fixed end. This value of n in (98) gives,

$$\Sigma Px = Pl \times 0.174$$

Equation (99) has two values of n, but the other is not within the limits of the problem.

The position of the weight, which will give a maximum strain at the fixed end, is found by making (97) a maximum. Proceeding in the usual way, we find:—

$$n = 1 \pm \tfrac{1}{3}\sqrt{3} = 0.422 + \ldots\ldots\ldots\ldots\ldots\ldots(101)$$

which in Eq. (97) gives, $\Sigma Px = Pl \times 0.181 \ldots\ldots\ldots(102)$

and in Eq. (98) $\Sigma Px = Pl \times 0.131+.$

To find where P must be applied so that the strain at the point of application will equal the strain at the fixed end, make Equations (97) and (98) equal to each other, and find n. This gives,

$$n = \begin{cases} 1. \\ 3.4141 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(103) \\ 0.5858 +. \end{cases}$$

But $n = 0.5858 +$ is the only practical value.

To find where P must be applied so that the curve at that point shall be horizontal, make $\frac{dy}{dx} = 0$, and $x = nl$ in (95).

$$\text{This gives } n = \begin{cases} 1. \\ 3.4141 \\ 0.5858 \end{cases}$$

which are the same as the preceding values of n. To find the corresponding deflection, make $x = nl$, and $n = 0.5858 +$, in (93), and we find

$$\Delta = 0.0098 \frac{Pl^3}{EI} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(104)$$

$$\left.\begin{aligned} \text{For } n &< 0.5858, \text{ tang } i \text{ is } + \\ n &> 0.5858, \text{ tang } i \text{ is } - \\ n &= 0.5858, \text{ tang } i \text{ is } 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots(105)$$

To find the maximum deflection when $n = 0.634$, make $\frac{dy}{dx} = 0$ in Eq. (92) or (95), according as the greater deflection is to the right or left of P. But, according to Eq. (105), it belongs to the curve AD; hence use Eq. (92). Making $n = 0.634$ in Eq. (92), placing it equal zero, and solving gives,

$$x = 0.6045l;$$

which in Eq. (93) gives,

$$y = \Delta = 0.00957 \frac{Pl^3}{EI} \dots\dots\dots\dots\dots (106)$$

To find where P must be applied so as to give an absolute maximum deflection; first find the abscissa of the point of maximum deflection, when P is applied at any point by making $\frac{dy}{dx} = 0$ in Eq. (92), and thus find

$$x = \frac{2(3 - n)n^2 - 4n}{(3 - n)n^2 - 2} l \dots\dots\dots (107)$$

which, substituted in Eq. (93) gives the corresponding maximum deflection. Then find that value of n which will make the expression a maximum.

To find the deflection when P is placed at the middle, make $n = \frac{1}{2}$ in Eq. (93) or Eq. (96), which gives

$$\delta = \frac{7}{48.16} \frac{Pl^3}{EI}.$$

The point of contra-flexure in the curve AD is found by making $\frac{d^2y}{dx^2} = 0$ in (91) (see Dif. Cal.) which gives,

$$x = \frac{3n^2 - n^3 - 2n}{3n^2 - n^3 - 2} l$$

If $n = \frac{1}{2}$, $x = \frac{3}{11} l$.

The second member of Eq. (91) is the moment of applied forces, and as it is nought at the point of contra-flexure, it follows that at that point there is no bending stress, and hence no elongation or compression of the fibres, but only a transverse shearing stress. The value of the transverse shearing is

$$Ss = EI \frac{d^3y}{dx^3} = -P + \tfrac{1}{2}n^2 (3 - n) P$$

which compared with Eq. (90) shows that the shearing strain at any point of the curve AD subtracted from the reaction at B equals the total load P.

If a beam rests upon three horizontal equidistant supports, and two weights, each equal P, are placed upon it, one on each side of the central support and equidistant from it, it fulfils the condition of a beam fixed at one end and supported at the other, as before stated, and the amount which each support will sustain for incipient flexure may easily be found from the preceding equations.

The three supports will sustain $2P$, and the end supports each sustain $V = \frac{1}{2}n^2(3 - n)P$. (See Eq. (90).)

Hence, the central support sustains

$$V' = 2P - n^2(3 - n)P.$$

If $n = \frac{1}{2}$, $V = \frac{5}{16} P$, and $V' = \frac{22}{16} P$.

106. CASE VIII.—LET THE BEAM BE FIXED AT ONE END, SUPPORTED AT THE OTHER, AND UNIFORMLY LOADED OVER ITS WHOLE LENGTH,

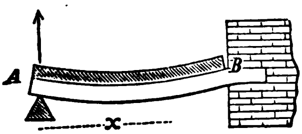

FIG. 45.

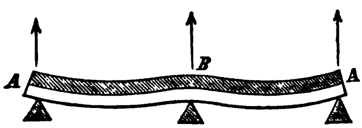

FIG. 46.

Take the origin at A, Figs. 45 and 46, and the notation the same as in the preceding cases, then Equation (50) becomes

$$EI\frac{d^2y}{dx^2} = \frac{1}{2}wx^2 - Vx \qquad (108)$$

Integrating gives $\frac{dy}{dx} = \frac{w}{6EI}(x^3 - l^3) + \frac{V}{2EI}(l^2 - x^2)$, (109)

$$\text{and } y = \frac{w}{24EI}(x^4 - 4l^3 x) + \frac{V}{6EI}(3l^2x - x^3) \qquad (110)$$

in which $\frac{dy}{dx} = 0$ for $x = l$, and $y = 0$ for $x = 0$.

If $V = 0$, these equations become the same as those under CASE II.

In Equation (110) y is also zero, for $x = l$; for which values we have $V = \frac{3}{8} W = \frac{3}{8}wl$ . . . .(111)

This value substituted in Equations (108), (109), and (110) gives:—

$$EI\frac{d^2y}{dx^2} = \tfrac{1}{8}wx(4x - 3l) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (112)$$

$$\frac{dy}{dx} = \frac{w}{48EI}(8x^3 - 9lx^2 + l^3) \ldots\ldots\ldots\ldots\ldots\ldots (113)$$

$$y = \frac{w}{48EI}(2x^4 - 3lx^3 + l^3x) \ldots\ldots\ldots\ldots\ldots\ldots (114)$$

The point of maximum deflection is found by placing Equation (113) equal zero and solving for x. This gives

$$x = l,$$

$$\text{and}, x = \frac{1 \pm \sqrt{33}}{16}l = 0.4215l,$$

using the positive value only; and this in Eq. (114) gives

$$y = \Delta = 0.0054\frac{Wl^3}{EI} \ldots\ldots\ldots\ldots\ldots\ldots (115)$$

There are two maxima strains; one for $x = l$; the other for $x = \frac{3}{8}l$. The former in (112) gives

$$\Sigma Px = \tfrac{1}{8}wl^2 = \tfrac{1}{8}Wl \ldots\ldots\ldots\ldots\ldots\ldots (116)$$

and the latter gives

$$\Sigma Px = -\tfrac{9}{128}Wl = -\tfrac{1}{14}Wl \text{ nearly.}$$

The point of contra-flexure is found from Equation (112) to be at $x = \frac{3}{4}l$, at which point the longitudinal strains are zero, and there is only transverse shearing.

From Eq. (112) we have

$$Ss = EI\frac{d^3y}{dx^3} = wx - \tfrac{3}{8}wl$$

For $x = \frac{3}{4}l$ we have $Ss = \frac{3}{8}wl \ldots\ldots\ldots\ldots\ldots\ldots (116a)$

If the beam is supported by three props, which are in the same horizontal, Fig. 46, then each part is subjected to the same conditions as the single beam in Fig. 45. Hence, if W is the load on half the beam, each of the end props will sustain $V = \frac{3}{8}W$ (Eq. (111)), and the middle prop will sustain $2W - \frac{3}{8}W = \frac{5}{4}W$.

From the supported end, A, to the point of contra-flexure

($\frac{3}{4}l$) the beam is in the same condition as a beam which is supported at its ends and uniformly loaded. Hence the supported end sustains $\frac{1}{2}$ of $\frac{3}{4}wl = \frac{3}{8}wl$, as before found. The shearing strain at the point of contra-flexure must be the same as at the supported end, which agrees with Eq. (116a).

Such are the teachings of the "Common theory." But the mathematical conditions here imposed are never realized. It is impossible to maintain the props exactly in the same horizontal. As they are elastic they will be compressed, and as the central one will be most compressed, the tendency will be to relieve the strain on it and throw a greater strain upon the end supports. If the supports be maintained in the same horizontal, the results above deduced will be practically true for *very small* deflections, within the elastic limits.

107. CASE IX.—LET THE BEAM BE FIXED AT BOTH ENDS AND A WEIGHT REST UPON IT AT ANY POINT.

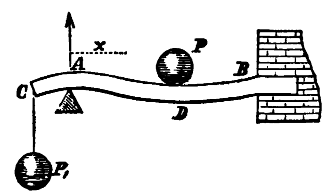

FIG. 47.

To simplify the case, suppose that the weight rests at the middle of the length.

Let the beam be extended over one support and a weight, P_1 rest at C, sufficient to make the curve horizontal over the support A. We have $V = P_1 + \frac{1}{2}P$.

Let $AC = ql$.

Then for the curve AD we have,

$$EI\frac{d^2y}{dx^2} = P_1(ql + x) - Vx = P_1 ql - \tfrac{1}{2}Px$$

$$\therefore EI\frac{dy}{dx} = P_1 qlx - \tfrac{1}{4}Px^2 + (C_1 = 0).$$

To find P_1 observe that $\frac{dy}{dx} = 0$ for $x = \frac{1}{2}l$;

$$\therefore 0 = \tfrac{1}{2}P_1 ql^2 - \tfrac{1}{16}Pl^2; \therefore P_1 q = \tfrac{1}{8}P.$$

This reduces the preceding equations to the following:—

$$EI\frac{d^2y}{dx^2} = \frac{1}{8}P(l - 4x) \ldots\ldots\ldots\ldots(117)$$

$$\frac{dy}{dx} = \frac{P}{8EI}(lx - 2x^2) \ldots\ldots\ldots\ldots(110)$$

and by integrating again, we find:—

$$y = \frac{P}{48EI}(3lx^2 - 4x^3) \ldots\ldots\ldots\ldots(119)$$

$$\text{For } x = \tfrac{1}{2}l \text{ in (119), } y = \Delta = \frac{Pl^3}{192EI} \ldots\ldots\ldots(120)$$

There is no algebraic maximum of the moment of strain as given in the second member of Equation (117), but inspection shows that within the limits of the problem the moment is greatest for $x = 0$ or $x = \frac{1}{2}l$. These in (117) give the same value, with contrary signs; hence the moment of greatest strain is

$$\Sigma Px = \pm \tfrac{1}{8}Pl \ldots\ldots\ldots\ldots(121)$$

The moment is zero for $x = \frac{1}{4}l$.

108. CASE X.—LET THE BEAM BE FIXED AT BOTH ENDS AND A LOAD UNIFORMLY DISTRIBUTED OVER ITS WHOLE LENGTH.

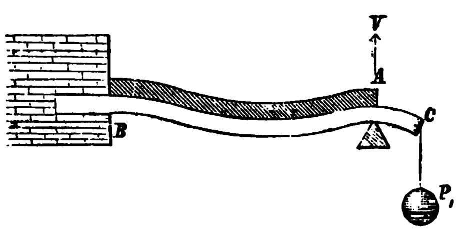

FIG. 48.

e notation being the same as before used, we have

$$V = P_1 + \tfrac{1}{2}wl.$$

et $ql = AC.$

he equation of moments is

$$EI\frac{d^2y}{dx^2} = \tfrac{1}{2}wx^2 - Vx + P_1(ql + x)$$

$$= \tfrac{1}{2}wx^2 - \tfrac{1}{2}wlx + P_1ql.$$

Integrating, and observing that $\frac{dy}{dx} = 0$ for $x = 0$; also $y = 0$ for $x = 0$, and we have

$$EI\frac{dy}{dx} = \tfrac{1}{6}wx^3 - \tfrac{1}{4}wlx^2 + P_1qlx$$

$$EIy = \tfrac{1}{24}wx^4 - \tfrac{1}{12}wlx^3 + \tfrac{1}{2}P_1qlx^2.$$

But $\frac{dy}{dx} = 0$ for $x = l$; also $y = 0$ for $x = l$;

$$\therefore P_1 = \frac{1}{12}\frac{wl}{q} = \frac{W}{12q}$$

which substituted in the previous equations give:—

$$EI\frac{d^2y}{dx^2} = \frac{W}{12l}\Big[l^2 - 6x(l-x)\Big] \ldots\ldots\ldots (122)$$

$$\frac{dy}{dx} = \frac{W}{12EIl}\Big[(l-x)(l-2x)x\Big] \ldots\ldots\ldots (123)$$

$$y = \frac{W}{24EIl}(l-x)^2x^2 \ldots\ldots\ldots (124)$$

For $x = \tfrac{1}{2}l$ in (124), $y = \Delta = \frac{1}{384}\frac{Wl^3}{EI} \ldots\ldots\ldots (125)$

Making $\frac{d^2y}{dx^2} = 0$ we find for the points of contra-flexure

$$x = \begin{cases} 0.7887l \\ 0.2113l \end{cases}$$

at which point there is no longitudinal strain, but a transverse shearing strain. We have

$$Ss = \frac{d\left(EI\frac{d^2y}{dx^2}\right)}{dx} = \frac{W}{2l}[2x - l]$$

which is equal to $\tfrac{1}{2}W$ at the ends (either $\pm$) and zero at the middle. At the first point of contra-flexure ($x = 0.2113l$) the shearing strain is $0.2887\,W$, to which add the load on that part, $= 0.2113\,W$, gives $0.5000\,W$, or $\frac{1}{2}$ the total load.

The maximum moments are for $x = 0$ and $x = \frac{1}{2}l$.

For $x = 0$, the second member of Eq. (112) gives $\frac{1}{12}Wl$. (126)

For $x = \frac{1}{2}l$, " " " $-\frac{1}{24}Wl$.

Hence the greatest strain is over the support, at which point it is twice as great as at the middle. If $W = P$, we see that the strain over the support is $\frac{2}{3}$ as great in this case as in the former.

109. RESULTS COLLECTED.

No. of the case.	Condition of the beam.	How loaded.	General moment of flexure.	Maximum moment of stress.	Relative moment.	Relative max. deflection or co-efficient of $\frac{l^3}{EI}$
I.	Fixed at one end.	Load at free end.	Px. *Eq.* (53).	Pl	24	$\frac{1}{3}P$. *Eq.* (57).
II.		Uniform load.	$\frac{1}{2}wx^2$ *Eq.* (58).	$\frac{1}{2}Wl$	12	$\frac{1}{8}W$. *Eq.* (61).
IV.	Supported at the ends.	At the middle.	$\frac{1}{2}Px$. *Eq.* (71).	$\frac{1}{4}Pl$.	6	$\frac{1}{48}P$. *Eq.* (73).
V.		Uniform.	$\frac{1}{2}w(lx-x^2)$. *Eq.* (74).	$\frac{1}{8}Wl$.	3	$\frac{5}{384}W$ *Eq.* (76).
VII.	Fixed at one end and supported at the other.	At 0.634l from fixed end *Eq.* (100).	For AD *Eq.* (91). For DB *Eq.* (94).	$\frac{1}{2}(2\sqrt{3}-3)Pl$.	4+	$\frac{P}{102}$ nearly. *Eq.* (104).
VIII.		Uniform.	$\frac{1}{8}w(4x^2-3lx)$. *Eq.* (112).	$\frac{1}{8}Wl$. *Eq.* (116).	3	$\frac{W}{185}$ nearly. *Eq.* (115).
IX.	Fixed at both ends.	At the middle.	$\frac{1}{8}P(l-4x)$. *Eq.* (117).	$\frac{1}{8}Pl$. *Eq.* (121).	3	$\frac{P}{192}$. *Eq.* (120).
X.		Uniform.	$\frac{W}{12l}(l^2-6lx+6x^2)$ *Eq.* (122).	$\frac{1}{12}Wl$. *Eq*, (126).	2	$\frac{W}{384}$ *Eq*, (125).

110. REMARKS.—It will be seen that the greatest strains in the 1st and 2d cases are as 2 to 1; and the same ratio holds in the 4th and 5th cases; but in the 9th and 10th the ratio is as 3 to 2. The maximum strains in Cases VII. and VIII. do not occur at the points of maximum deflection.

Although the moment in the first case is to that in the 2d as 2 to 1, yet the deflections are as 8 to 3; and in the 4th and 5th cases the deflections are as 8 to 5.

A comparison of Cases IV. and IX. shows the advantage of fixing the ends of the beam. The same remark applies to Cases V. and X. In the former cases the strain is only one-half as great when the beam is fixed at the ends as when it is supported, and in the latter two-thirds as great.

Other interesting results may be seen by examining the table.

The following are the results of some experiments made by James B. Francis:

Experiment 1. A bar of "common English refined" iron, marked "J crown K, best," 12 feet 2¾ inches long, mean width 1.535 inch, mean depth 0.367 inch, was laid on the 4 bearings, and loaded at the centre of each span, so as to make the deflections the same, the weight at the middle span being 82.84 pounds, and at each of the end spans 52.00 pounds. The deflections with these weights were as follows:

At the centre of the middle span.............................0.281 inches.
At the centre of the end spans....0.275 and 0.284 inches, mean, 0.280 "

The deflections of the 3 spans being, as nearly as practicable, the same, the middle span is in the condition of a beam "fixed at both ends and loaded in the middle," each of the end spans "being fixed at one end and supported at the other." A piece 3 feet 11¼ inches long was then cut off from each end of the bar, leaving a bar 4 feet 4¾ inches long, which was replaced in its former position and loaded with the same weight (82.84 pounds) as before, when its deflection was found to be 1.059 inch, or 3.77 times the deflection when "fixed at both ends and loaded in the middle."

Experiment 2. A bar of iron of the same quality and length as in Experiment 1, nearly square, its mean width being 0.553 inch, and mean depth 0.549 inch, was laid on the same bearings, and loaded with the same weights, the deflections being as follows:

At the centre of the middle span.............................. 0.342 inch.
At the centre of the end spans.......0.238 and 0.244 inch, mean, 0.241 "

The bar was then reduced in length as in Experiment 1, leaving 4 feet 3¾ inches, which was replaced in its former position and loaded with the same weight (82.84 pounds) as before, when its deflection was found to be 0.983 inch, or 4.06 times the deflection, "when fixed at both ends and loaded in the middle."

The result of both experiments agreed substantially with the deflection in

the case of a beam "fixed at one end, supported at the other, and loaded in the middle," which is $\frac{7}{16}$ = 0.438 of the deflection in the case, "supported at each end and loaded in the middle." In the foregoing experiments, the end spans correspond to this case, and the observed deflections with a weight of 52 pounds, were 0.419 and 0.391 respectively, of the deflections in the case, "supported at the end and loaded in the middle," differing somewhat, but not very widely, from the proportion given above.

111. PROBLEM.—A PRISMATIC BEAM RESTS ON A SUPPORT AT THE MIDDLE OF ITS LENGTH, AND BARELY COMES IN CONTACT WITH SUPPORTS WHICH ARE PLACED AT EACH END. SUPPOSE THAT AN UNIFORM LOAD IS PLACED ON ONE-HALF OF THE BEAM; IT IS REQUIRED TO FIND THE WEIGHT P WHICH, IF PLACED AT THE END WHICH IS REMOTE FROM THE UNIFORM LOAD, WILL CAUSE THE END TO WHICH IT IS APPLIED TO REMAIN IN CONTACT WITH THE SUPPORT.

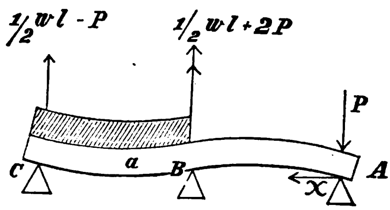

FIG. 49.

In Fig. 49,

Let $l = AB = BC$;

w = the load per foot of length on BC; and

P = the weight at A which is necessary in order to keep the end down to the support.

Take the origin at A, x horizontal and y vertical.

Since no part of P is supported by A, it must be balanced by a part of the reaction of the support at C.

The supports B and C each sustain one-half the uniform load; hence,

$\frac{1}{2}wl - P$ will be the reaction of the support C;

$\frac{1}{2}wl + 2P$ will be the reaction of the support B.

First consider the curve BC, and while so doing suppose that the part AB is rigid; in other words, that the weight P does not cause AB to bend while the part BC is elastic. We then have for any point a, between B and C.

Px = the moment of P;

$(\frac{1}{2}wl + 2P)(x - l)$ = the moment of the reaction at B, which will have an opposite sign to Px;

$w(x-l)$ = the load on Ba;

$\frac{1}{2}w(x-l)^2$ = the moment of the load on Ba.

Hence,

$$(a) \quad EI\frac{d^2y}{dx^2} = Px + \tfrac{1}{2}w(x-l)^2 - (\tfrac{1}{2}wl + 2P)(x-l)$$

$$\therefore EI\frac{dy}{dx} = \tfrac{1}{2}Px^2 + \tfrac{1}{6}w(x-l)^3 - \tfrac{1}{4}(wl + 4P)(x-l)^2 + C_1$$

also, $EIy = \frac{1}{6}Px^3 + \frac{1}{24}w(x-l)^4 - \frac{1}{12}(wl + 4P)(x-l)^3 + C_1x + C_2$

But $y = 0$ for $x = l$; and

$y = 0$ for $x = 2l$; which values in the last equation give

$$0 = \tfrac{1}{6}Pl^3 + C_1 l + C_2$$

$$0 = \tfrac{4}{3}Pl^3 + \tfrac{1}{24}wl^4 - \tfrac{1}{12}(wl + 4P)l^3 + 2C_1 l + C_2 = Pl^3 - \tfrac{1}{24}wl^4 + 2C_1 l + C_2.$$

Eliminating successively C_1 and C_2 from these equations and we have,

$$C_1 = -\tfrac{5}{6}Pl^2 + \tfrac{1}{24}wl^3.$$

$$C_2 = \tfrac{2}{3}Pl^3 - \tfrac{1}{24}wl^4.$$

These substituted in the preceding equations give

$$(b) \quad EI\frac{dy}{dx} = \tfrac{1}{2}Px^2 + (\tfrac{1}{6}wx - \tfrac{5}{12}wl - P)(x-l)^2 - \tfrac{5}{6}Pl^2 + \tfrac{1}{24}wl^3$$

$$(c) \quad EIy = \tfrac{1}{6}Px^3 + (\tfrac{1}{24}wx - \tfrac{1}{8}wl - \tfrac{1}{3}P)(x-l)^3 - \tfrac{5}{6}Pl^2x + \tfrac{1}{24}wl^3x + \tfrac{2}{3}Pl^3 - \tfrac{1}{24}wl^4.$$

These equations will enable us to determine all the properties of the curve BC. But the solution of the problem only makes it necessary to find the inclination at B. For this $x=l$ and we find

$$(d)\quad \frac{dy}{dx} = \left(-\tfrac{1}{3}P + \tfrac{1}{24}wl\right)\frac{l^2}{EI}$$

Now consider AB as flexible, and we have

$$(e)\quad EI\frac{d^2y}{dx^2} = Px$$

$$\therefore \frac{dy}{dx} = \frac{Px^2}{2EI} + C_3.$$

But this value of the tangent when $x = l$ is the same as the preceding value

$$\therefore C_3 = (-20P + wl)\frac{l^2}{24EI}$$

$$(f)\quad \therefore \frac{dy}{dx} = (12Px^2 - 20Pl^2 + wl^3)\frac{1}{24EI}.$$

Integrating again gives

$$y = (4Px^3 - 20Pl^2x + wl^3x)\frac{1}{24EI} + C_4$$

But by the conditions of the problem

$$y = 0 \text{ for } x = 0 \therefore C_4 = 0$$

Also, $y = 0$ for $x = l$

$$\therefore 0 = -16Pl^3 + wl^4,$$

$$\text{or } P = \tfrac{1}{16}wl = \tfrac{1}{16}W.$$

This problem was suggested by the conditions of the draw-bridge. If the end is not held down, the distance which it will raise by an uniform load on the other half is found from Eq. (d), by making $P = 0$, and multiplying by l. This will give

$$\frac{wl^4}{24EI}$$

112. TO FIND THE REACTION OF THE SUPPORTS, we may first find the bending moments over the supports, according to Clapeyron's method.

FIG. 49a.

Let A, B and C be any three consecutive supports;
l, the segment AB, and l' the segment BC;
w the pressure on a unit of length on AB, and w' on BC;
X_1, X_2 X_3 the bending moments at A, B, and C, respectively;

Take the origin at A. The moment of external forces upon any point in the segment AB, will involve the moment of all the reactions at the left of A, and the moment of the total load to the left of A, plus the moment of the load on x. The moments of the two former may contain the first power of x, and possibly they may also contain a constant. Let A and B be constants; then the equation for the moment of flexure will be

$$(a)\quad EI\frac{d^2y}{dx^2} = A + Bx - \tfrac{1}{2}wx^2$$

If $x = 0$, the second member becomes

$$A = X_1,$$

and if $x = l$, we have $A + Bl - \frac{1}{2}wl^2 = X_2$

$$\therefore B = \frac{1}{2}wl + \frac{X_2 - X_1}{l}$$

$$(b)\quad \therefore EI\frac{d^2y}{dx^2} = X_1 + \frac{X_2 - X_1}{l}x + \tfrac{1}{2}wlx - \tfrac{1}{2}wx^2$$

similarly, if the origin be taken at B we have

$$(c)\quad EI\frac{d^2y}{dx^2} = X_2 + \frac{X_3 - X_2}{l} + \tfrac{1}{2}w'lx - \tfrac{1}{2}w'x^2.$$

Integrating Eq. (b), observing that $\frac{dy}{dx} = \text{tang } i'$ for $x = 0$, and tang i'' for $x = l$ and $y = 0$ for $x = 0$ and for $x = l$. Between the equations thus formed eliminate tang i', and find the value of tang i'' from Eq. (c), and substitute its value in the preceding. This done and the result may be reduced to the following form:—

$$(d)\quad X_1 l + 2X_2(l + l') + X_3 l' + \tfrac{1}{4}(wl^3 + w'l'^3) = 0$$

which expresses the relation between the bending moments at any three consecutive points of support.

By applying this equation successively to the successive points, the bending moments at all the points of support may be found, after which the bending moments of any point of any of the segments by Eq. (a) or (b). The reactions may also be easily found by the aid of the results.

If $l = l' =$ &c., and $w = w' =$ &c., Eq. (d) becomes

$$(e)\quad X_1 + 4X_2 + X_3 + \tfrac{1}{2}wl^2 = 0,$$

and for the second, third, and fourth supports we have

$$X_2 + 4X_3 + X_4 + \tfrac{1}{2}wl^2 = 0,$$

and so on. By taking the difference between these, we find the relation between the bending moments for four consecutive points of support independent of the uniform load.

Example.—Suppose that there are five points of support, equidistant; and the load uniform.

The bending moment at the first support, A, Fig. 49*a*, is zero; that at B equals that at D,—supposing that there is a fifth point, E, beyond D. Hence for the first two segments, Eq. (*e*), gives

$$0+4X_2+X_3+\tfrac{1}{2}wl^2=0$$

and for the second and third segments;

$$X_2+4X_3+X_2+\tfrac{1}{2}wl^2=0,$$

$$\text{or } X_2+2X_3+\tfrac{1}{4}wl^2=0;$$

and by elimination

$$X_2=-\tfrac{3}{28}wl^2, \text{ and } X_3=-\tfrac{1}{14}wl^2.$$

Let P_1, P_2, &c., be the reactions of the supports at A, B, &c., then the moments at B are

$$P_1\,l-\tfrac{1}{2}wl^2=X_2=-\tfrac{3}{28}wl^2$$

$$\therefore P_1=\tfrac{11}{28}W,$$

where $W=$ the load on each segment.

For the moments at C we have

$$P_1.2l+P_2l-wl.\tfrac{3}{2}l-wl.\tfrac{1}{2}l=X_3=-\tfrac{1}{14}wl^2$$

$$\therefore P_2=\tfrac{32}{28}wl.$$

The total load is

$$2P_1+2P_2+P_3=4wl$$

$$\therefore P_3=\tfrac{26}{28}wl,$$

which are the same as those given in the table on page 135.

Whatever be the number of props, we have for the first segment

$$EI\frac{d^2y}{dx^2}=-V_1x+\tfrac{1}{2}wx^2$$

$$EI\frac{dy}{dx}=-\tfrac{1}{2}V_1x^2+\tfrac{1}{6}wx^3+C_1$$

$$EIy=-\tfrac{1}{6}V_1x^3+\tfrac{1}{24}wx^4+C_1x+C_2$$

For $x=0$, $y=0$; also for $x=l$, $y=0$

$$\therefore C_2=0, \text{ and } C_1=\tfrac{1}{6}V_1l^2-\tfrac{1}{24}wl^3.$$

Hence over the first support we have

$$\frac{dy}{dx}=\text{tang } i=(4V_2-W)\frac{l^2}{24EI}$$

and for the deflection at the middle of the first segment

$$\Delta=(24V_1-7W)\frac{l^3}{384EI};$$

which is always somewhat less than the maximum deflection, except when the beam is supported at its ends only.

ADDITIONAL PROBLEMS WHICH ARE PURPOSELY LEFT UNSOLVED.

1. Suppose that a beam is supported at its extremities, and has two forces at any point between. In this case the curve between the support and the nearest force will have one equation; the curve between the forces another; and the remaining part a third.

2. In the preceding case, if the forces are equal and equidistant from the supports, the curve between the forces will be the arc of a circle.

3. Suppose that the beam is uniformly loaded and rests on four supports.

4. Suppose that the beam is supported at its extremities and has a load uniformly increasing from one support to the other.

5. Suppose that the beam is uniformly loaded over any portion of its length.

6. Suppose that it has forces applied at various points.

These problems will suggest many others.

7. Suppose that a beam is supported at several points, and loaded uniformly over its whole length.

Let $W=$ the weight between each pair of supports,
V_1, V_2, V_3, &c., be the reactions of the supports, counting from one end,
and let the distances between the supports be equal.

Then we have:—

No. of Supports.	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}
2	$\frac{1}{2}W$	$\frac{1}{2}W$	Frac	tional	parts		of W.			
3	$\frac{3}{8}$	$\frac{10}{8}$	$\frac{3}{8}$							
4	$\frac{4}{10}$	$\frac{11}{10}$	$\frac{11}{10}$	$\frac{4}{10}$						
5	$\frac{11}{28}$	$\frac{32}{28}$	$\frac{26}{28}$	$\frac{32}{28}$	$\frac{11}{28}$					
6	$\frac{15}{38}$	$\frac{43}{38}$	$\frac{37}{38}$	$\frac{37}{38}$	$\frac{43}{38}$	$\frac{15}{38}$				
7	$\frac{41}{104}$	$\frac{118}{104}$	$\frac{108}{104}$	$\frac{106}{104}$	$\frac{108}{104}$	$\frac{118}{104}$	$\frac{41}{194}$			
8	$\frac{56}{142}$	$\frac{161}{142}$	$\frac{137}{142}$	$\frac{143}{142}$	$\frac{143}{142}$	$\frac{137}{142}$	$\frac{161}{142}$	$\frac{56}{102}$		
9	$\frac{153}{388}$	$\frac{440}{388}$	$\frac{374}{388}$	$\frac{392}{378}$	$\frac{386}{388}$	$\frac{392}{388}$	$\frac{374}{388}$	$\frac{440}{388}$	$\frac{153}{388}$	
10	$\frac{209}{530}$	$\frac{601}{530}$	$\frac{511}{530}$	$\frac{535}{530}$	$\frac{529}{530}$	$\frac{529}{530}$	$\frac{535}{530}$	$\frac{511}{530}$	$\frac{601}{530}$	$\frac{209}{530}$

If the beams and props were perfectly rigid, all but the end ones would sustain W, and the end ones each $\frac{1}{2}$ W.

113. BEAMS OF VARIABLE SECTIONS.

For these I is variable, and its value must be substituted in Equation (50) before the integration can be performed. As an example, let the beam be fixed at one extremity, and a weight P, be suspended at the free extremity, Fig. 50. Let the breadth be constant, and the longitudinal vertical section be a parabola. Then all the transverse sections will be rectangles.

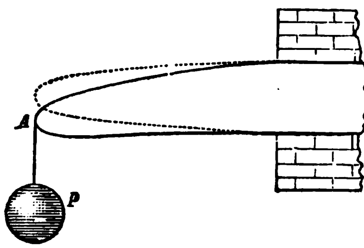

FIG. 50.

Let $l =$ the length,
$b =$ the breadth, and
$d =$ the depth at the fixed extremity.

If y is the whole variable depth at any point, we have, from the equation of the parabola,

$(\frac{1}{2}y)^2 = px$, or $\frac{1}{4}d^2 = pl$, $\therefore p = \frac{d^2}{4l}$, in which p is the parameter of the parabola.

$$\therefore y^2 = \frac{d^2x}{l} \qquad (127)$$

From Equation (51) we have

$I = \frac{1}{12}by^3$, in which substitute y, from Equation (127), and we have $I = \frac{1}{12}\frac{bd^3}{l^{\frac{3}{2}}}x^{\frac{3}{2}}$ ....(128)

The equation of moments is, see Equation (50),

$EI\frac{d^2y}{dx^2} = Px$, in which substitute I, from Equation (128), and we have

$$\frac{d^2y}{dx^2} = \frac{12Pl^{\frac{3}{2}}}{Ebd^3}x^{-\frac{1}{2}}$$

Multiply by dx and integrate, observing that $\frac{dy}{dx} = 0$ for $x = l$

and we have

$$\frac{dy}{dx}=\frac{24Pl^{\frac{3}{2}}}{Ebd^3}(x^{\frac{1}{2}}-l^{\frac{1}{2}})$$

Integrating again gives

$$y=\frac{8Pl^{\frac{3}{2}}}{Ebd^3}(2x^{\frac{3}{2}}-3l^{\frac{1}{2}}x)$$

$$y \text{ is zero for } x=0.$$
$$y=\Delta \text{ for } x=l;$$

$$\therefore \Delta=-\frac{8Pl^3}{Ebd^3}\ldots\ldots\ldots\ldots\ldots\ldots(129)$$

If, in Equation (57), we substitute $I=\frac{1}{12}bd^3$ (Eq. (51)), it becomes

$$\Delta=-\frac{4Pl^3}{Ebd^3}$$

which is one-half that of Eq. (129); hence the deflection of a prismatic beam is one-half that of a parabolic beam of the same length, breadth, and greatest depth, when fixed at one end and free at the other, and has the same weight suspended at the free end.

In a similar manner the equation of the curve may be found for any other form of beam, if the law of increase or decrease of section is known. Several examples may be made of beams of uniform strength, which will be given in Chapter VII.

114. BEAMS SUBJECTED TO OBLIQUE STRAINS.

Let the beam be prismatic, fixed at one end, and support a weight, P, at the free end; the beam being so inclined that the direction of the force shall make an obtuse angle with the axis of the beam, as in Fig. 51.

Let $P_1=P\sin\theta=$ component of P perpendicular to the axis of the beam, and

$P_2=P\cos\theta=$ component parallel to the axis of the beam.

Take the origin at the free end, the axis of x being parallel to the axis of the beam, and y perpendicular to it.

Then Equation (50) becomes

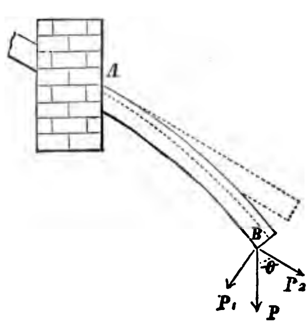

FIG. 51.

$$EI\frac{d^2y}{dx^2} = -P_1x + P_2y$$

$$\text{or, } \frac{d^2y}{dx^2} = -p^2x + q^2y \quad \text{.........................} (130)$$

in which $p^2 = \frac{P_1}{EI}$; and $q^2 = \frac{P_2}{EI}$. The complete integral of (130) is (see Appendix).

$$y = C_1e^{qx} + C_2e^{-qx} + \frac{p^2}{q^2}x$$

The conditions of the problem give

$\frac{dy}{dx} = 0$ for $x = l$; and

$y = 0$ for $x = 0$; and these combined with the preceding equation give:—

$$0 = q\left(C_1e^{ql} - C_2e^{-ql}\right) + \frac{p^2}{q^2};$$

$$0 = C_1 + C_2;$$

From which C_1 and C_2 may be found, and the equation becomes completely known.

We also have $y = \Delta$ for $x = l$;

$$\therefore \Delta = C_1e^{ql} + C_2e^{-ql} + \frac{p^2}{q^2}l;$$

Next, suppose that the force makes an acute angle with the axis of the beam, as in Fig. 52.

For the sake of variety, take the origin at A, the fixed end, x, still coinciding with the axis of the beam before flexure. Using the same notation as in the preceding and other cases, we have

$$\frac{d^2y}{dx^2} = p^2(l-x) + q^2(\Delta - y) \quad \text{..........................} (131)$$

The complete integral is

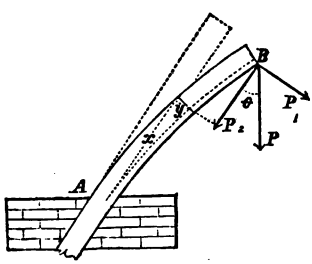

Fig. 52.

$$\Delta - y = A\sin q(x+B) - \frac{p^2}{q^2}(l-x) \quad (132)$$

in which A and B are arbitrary constants.

From the problem we have

$y = 0$ for $x = 0$;

$\frac{dy}{dx} = 0$ for $x = 0$; and

$y = \Delta$ for $x = l$;

by means of which the equation becomes completely known.

One difficulty in applying these cases in practice is in determining the value of I. Before it can be determined, the position of the axis must be known. According to Article 78, 3d case, it appears that the neutral axis does not coincide with the axis of the beam. Indeed, according to the same article, it is not parallel to the axis, and hence I is variable, and the equations above are only a secondary approximation; the first approximation being made in establishing Equation (50), and the next one in assuming I constant. In practice we assume that I is constant for prismatic beams, and that the neutral axis coincides with the axis of the beam.

115. FLEXURE OF COLUMNS.

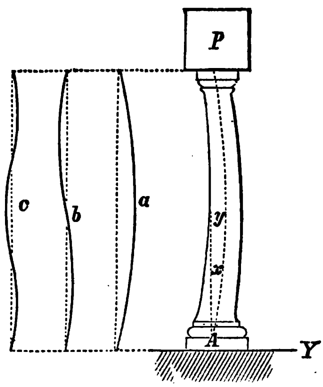

FIG. 54. FIG. 53.

If a weight rests upon the axis of a perfectly symmetrical and homogeneous column, we see no reason why it should bend; but in practice we know that it will bend, however symmetrical and homogeneous it may be, and however carefully the weight may be placed upon it. If the weight be small, the deflection may not be visible to the unaided eye. If the weight is not so heavy as to crush the column, an equilibrium will be established between the weight and the elastic resistance within the beam. Let the column rest upon a horizontal plane, and the weight P on the upper end be vertically over the lower end. Take the origin of coördinates at the lower end of the column, Fig. 53, x being vertical, and y horizontal. They must be so taken here, because x was assumed to coincide with the axis of the beam when Equation (50) was established. Then y being the ordinate to any point of the axis of the column after flexure, the moment of P is Py, which is negative in reference to the moment of resisting forces, because the curve is concave to the axis of x, in which case the ordinate and second differential coefficient must have contrary signs (Dif. Cal.). Hence we have,

$$EI\frac{d^2y}{dx^2} = -Py \qquad (133)$$

Multiply by dy and integrate (observing that dx is constant), and find

$$\frac{dy^2}{dx^2} = -\frac{P}{EI}y^2 + C_1.$$

But $\frac{dy}{dx} = 0$ for $y = \Delta =$ the maximum deflection. These values in the preceding equation give $C_1 = \frac{P\Delta^2}{EI}$, which being substituted in the same equation and reduced gives

$$dx = \sqrt{\frac{EI}{P}} \times \frac{dy}{\sqrt{\Delta^2 - y^2}}$$

$$\therefore x = \sqrt{\frac{EI}{P}} \sin^{-1} \frac{y}{\Delta} + C_2.$$

But $y = 0$ for $x = 0 \therefore C_2 = 0$. Hence the preceding gives

$$y = \Delta \sin \sqrt{\frac{P}{EI}}x \ldots\ldots\ldots\ldots\ldots\ldots\ldots (134)$$

But $y = 0$ for $x = l$. Therefore, if n is an integer, these values reduce (134) to

$$\sqrt{\frac{P}{EI}} \times l = n\pi;$$

$$\therefore P = EI\frac{n^2\pi^2}{l^2} \ldots\ldots\ldots\ldots\ldots (135)$$

This value of P reduces (134) to

$$y = \Delta \sin n\pi \frac{x}{l}$$

which is the equation of the curve. It is dependent only upon the length of the column and the maximum deflection. If $n = 1$, the curve is represented by a, Fig. 54; if $n = 2$, by b; if $n = 3$, by c.

If $n = 1$, Equation (135) becomes

$$P = \frac{\pi^2}{l^2}EI \ldots\ldots\ldots\ldots\ldots\ldots\ldots (136)$$

which is the formula to be used in practice. We see that the resistance is independent of the deflection. If the column is cylindrical, $I = \frac{1}{4}\pi r^4$ (see Equation (52));

$$\therefore P = \frac{\pi^3 E}{4} \times \frac{r^4}{l^2} \dots\dots\dots\dots\dots\dots (137)$$

hence the resistance varies as the fourth power of the radius (or diameter), and inversely as the square of the length. If the column is square, $I = \frac{1}{12} b^4$ (Equation (51)),

$$\therefore P = \frac{\pi^2 E}{12} \times \frac{b^4}{l^2} \dots\dots\dots\dots\dots\dots (138)$$

These formulas, according to Navier* and Weisbach,† should be used only when the length is 20 times the diameter for cylindrical columns, or 20 times the least thickness for rectangular columns; and Navier says that for safety only $\frac{1}{10}$ of the calculated weight should be used in case of wood, and $\frac{1}{4}$ to $\frac{1}{6}$ in case of iron; but Weisbach says they should have a twenty-fold security.

Examples.—1. What must be the diameter of a cast-iron column, whose length is 12 feet, to sustain a weight of 30 tons (of 2,000 lbs. each); $E = 16,000,000$ lbs.; and factor of safety $\frac{1}{20}$. Ans. $d = 7.52$ in.

2. If the column be square and the data the same as in the preceding example, Equation (138) gives

$$b = \sqrt[4]{\frac{12 \times 60,000 \times (12 \times 12)^2 \times 20}{(3.1416)^2 \times 16,000,000}} = 6.6 \text{ inches.}$$

In the analysis of this problem I have followed the method of Navier; but practical men generally prefer the empyrical formulas of Article 62. But it will be observed that the *law* of strength, as given in the formulas in that article, are the same as those given in equations (137) and (138) for wooden columns, and nearly the same as for iron ones. The chief difference is in the coefficients, or constant factors. In the analysis it was assumed that the neutral axis coincides with the axis of the beam, but it is possible for the whole column to be compressed, although much more on the concave than on the con-

* Navier, *Résumé des Leçons*, 1838, p, 204.

† Weisbach's *Mechanics and Engineering*. Vol. 1, p. 219. 1st Am. ed.

vex side, in which case the neutral axis would be ideal, having its position entirely outside the beam on the convex side. In this case, if the ideal axis is parallel to the axis of the beam, the value of I will be constant; and equations (137) and (138) retain the same form. The problem of the *flexure* of columns is then more interesting as an analytical one than profitable as a practical one.

GRAPHICAL METHOD.

116. THE GRAPHICAL METHOD consists in representing quantities by geometrical magnitudes, and reasoning upon them, with or without the aid of algebraic symbols. This method has some advantage over purely analytical processes; for by it many problems which involve the spirit of the Differential and Integral Calculus may be solved without a knowledge of the processes used in those branches of mathematics; and in some of the more elementary problems, in which the spirit of the Calculus is not involved, the quantities may be directly presented to the eye, and hence the solutions may be more easily retained. It is distinguished, in this connection, from pure geometry by being applied to problems which involve mechanical principles, and to use it profitably in such cases requires a knowledge of the elementary principles of mechanics as well as of geometry.

But graphical methods are generally special, and often require peculiar treatment and much skill in their management. It is not so powerful a mode of analysis as the analytical one, and those who have sufficient knowledge of mathematics to use the latter will rarely resort to the former, unless it be to illustrate a principle or demonstrate a problem for those who cannot use the higher mathematics. A few examples will be given to illustrate this method.

117. GENERAL PROBLEM OF THE DEFLECTION OF BEAMS.—*To find the total deflection of a prismatic beam which is bent by a force acting normal to the axis of the beam* without the aid of the Calculus.

Let a beam AB, Fig. 55, be bent by a force, P, in which case the fibres on the convex side will be elongated, and those on the concave side will be compressed. Let AB be the neutral axis. Take two sections normal to the neutral axis at L

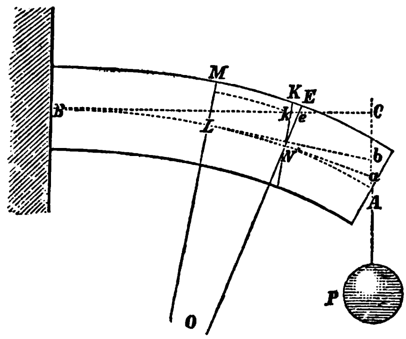

FIG. 55.

and N, which are *indefinitely near each other*. These, if prolonged, will meet at some point as O. Draw KN parallel to LO. Then will ke, $= \lambda$, be the distance between KN and EN at k, and is the elongation of the fibre at k. Let $eN = y$, then from the similar triangles kNe and LON we have

$$ON : Ne :: LN : ke = \lambda = \frac{Ne.\ LN}{ON} = \frac{LN}{ON} y.$$

If, now, we conceive that a force p, acting in the direction of the fibres, or, which is the same thing, acting parallel to the axis of the beam, is applied at k to elongate a single fibre, we have, from Equation (3) and the preceding one,

$$p = E\,\overline{\Delta a}\,\frac{ke}{LN} = \frac{E}{ON}\,y\,\overline{\Delta a},$$

in which $\overline{\Delta a}$ is the transverse section of the fibre. As the section NK turns about N on the neutral axis, the moment of this force is

$$py = \frac{E}{ON}\,y^2\,\overline{\Delta a}$$

which is found by multiplying the force by the perpendicular y.

This is the moment of a force which is sufficient to elongate or compress any fibre whose original length was LN, an amount equal to the distance between the planes KN and EN measured on the fibre or fibre prolonged. Hence, the sum of all the moments of the resisting forces is

$$\Sigma py = \frac{E}{ON} \Sigma y^2 \overline{\Delta a}$$

in which Σ denotes summation; and in the first member means that the sum of the moments of all the forces which elongate and compress the fibres is to be taken; and in the second member it means that the sum of all the quantities $y^2 \overline{\Delta a}$ included in the transverse section is to be taken. The quantity, $\Sigma y^2 \overline{\Delta a}$ is called the *moment of inertia*, which call I.

But the sum of the moments of the resisting forces equals the sum of the moments of the applied forces. Calling the latter ΣPX, in which X is the arm of the force P, and we have

$$\Sigma py = \Sigma PX = \frac{E}{ON} \Sigma y^2 \overline{\Delta a} = \frac{E.I}{ON}$$

$$\therefore ON = \frac{E.I}{\Sigma PX} \quad \text{.....................(139)}$$

In the figure draw Lb tangent to the neutral axis at L, and Na tangent at N. The distance ab, intercepted by those tangents on the vertical through A, is the deflection at A due to the curvature between L and N. As LN is indefinitely short, it may be considered a straight line, and equal x; and $Lb = LC$ very nearly for small deflections; and $LC = X$. (L stands for two points.)

By the triangles OLN and aLb, considered similar, we have

$$ON : x :: Lb : ab = \frac{Xx}{ON}$$

in which substitute ON from Equation (139) and we have

$$ab = \frac{Xx \Sigma PX}{E.I} \quad \text{................(140)}$$

which is sufficiently exact for small deflections. If, now, tangents be drawn at every point of the curve AB, they will divide the line AC into an infinite number of small parts, the sum of which will equal the line AC, the total deflection. But the expression for the value of each of these small spaces will be of the same form as that given above for ab, in which P, E, and I are constant.

This is as far as we can proceed with the general solution. We will now consider

PARTICULAR CASES.

118. CASE 1. LET THE BEAM BE FIXED AT ONE END, AND A LOAD, P, BE APPLIED AT THE FREE END.—This is a part of Case I., page 109, and Fig. 37 is applicable. The moment of P, in reference to any point on the axis, is PX. Hence ΣPX is simply PX, which, substituted in Equation (140), gives

$$ab = \frac{P}{EI} X^2 x$$

$$\therefore AC = \frac{P}{E.I} \Sigma X^2 x \quad \text{.............................(141)}$$

This equation has been deduced directly from the figure. It now remains to find the sum of all the values of X^2x, which result from giving to X all possible values from $X = 0$ to $X = l$.* To do this, construct a figure some property of which represents the expression, but which has not necessarily any other relation to the problem which is being solved. If X be used as a linear quantity, X^2 may be an area and X^2x will be a small volume. These conditions are represented by a pyramid, Fig. 56, in which

$AB = l =$ the altitude, and the base $BCDE$ is a square, whose sides, BC and CD, each $= l$. Let $bcde$ be a section parallel to the base, and make another section infinitely near it, and call the distance between the two sections x.

Then $Ab = X = bc = cd$,

$X^2 =$ area $bcde$, and

$X^2x =$ the volume of the lamina $bcde$,

which is the expression sought. The sum of all the laminæ of the pyramid which are parallel to the base is limited by the volume of the pyramid, and this equals the value of the expression ΣX^2x between the limits 0 and l. The volume of the pyramid is the area of the base ($= l^2$) multiplied by one-third the altitude ($\frac{1}{3}l$), or $\frac{1}{3}l^3$, which is the value sought.

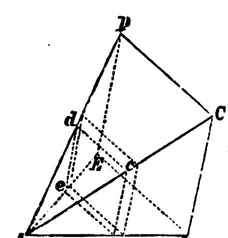

FIG. 56.

* This by the Calculus becomes $\int_0^l x^2 dx^2 = \frac{1}{3}l^3$.

Hence, $AO = \frac{Pl^3}{3E.I}$

which is the same as Equation (57).

The value of X^2x may also be found by statical moments as follows:—Let ABC, Fig. 57, be a triangle, whose thickness is unity, and which is acted upon by gravity (or any other system of parallel forces which is the same on each unit of the body). Take an infinitely thin strip, bc, perpendicular to the base, and let $AB = l = BC$,

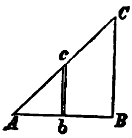

FIG. 57.

$Ab = X = bc$, and

$p =$ the weight of a unit of volume.

Then $Xx =$ the area of the infinitely thin strip bc, and

$pXx =$ the weight of the strip bc, and

$pX^2x =$ the moment of the strip, when A is taken as the origin of moments. If the weight of a unit of volume be taken as a unit, the moment becomes X^2x, which is the quantity sought, and the value of ΣX^2x from 0 to l is the moment of the whole triangle ABC. Its area is $\frac{1}{2}l^2$, and its centre of gravity $\frac{2}{3}l$ to the right of A. Hence the moment is $\frac{1}{3}l^3$ as before found.*

119. CASE II.—LET THE BEAM BE FIXED AT ONE END, AND UNIFORMLY LOADED OVER ITS WHOLE LENGTH.— This is the same as a part of Case II., page 101, and Fig. 39 is applicable.

Let X be measured from the free end, and

$w =$ the load on a unit of length; then

$wX =$ the load on a length X, and

$\frac{1}{2}X =$ the distance of the centre of gravity of the load from the section which is considered.

Hence the moment is $\frac{1}{2}wX^2$, which equals ΣPX, and Equation (140) becomes

$$ab = \frac{w}{2E.I}X^3x, \text{ and}$$

$$AC = \frac{w}{2E.I}\sum_{x=0}^{x=l} X^3x = \text{the total deflection.}$$

To find the value of ΣX^3x, observe, in Fig. 56, that X^2x is the volume of the lamina $bcde$, and this multiplied by the altitude of $A - bcde$, which is X, gives X^3x, the expression sought. Hence the sum sought is the *volume* of the pyramid $A - BCDE$, multiplied by the distance of the centre of gravity of the pyramid from the apex; or,

$$l^2 \times \tfrac{1}{3}l \times \tfrac{3}{4}l = \tfrac{1}{4}l^4$$

$$\therefore AC = \frac{wl^4}{8E.I} = \frac{Wl^3}{8E.I} \quad \text{(142)}$$

where W is the total load on the beam.

---

* This may be written $\sum_{x=0}^{x=l} X^2x = \frac{1}{3}l^3$.

120. CASE III.—LET THE BEAM BE SUPPORTED AT ITS ENDS AND LOADED AT THE MIDDLE BY A WEIGHT P, as in Fig. 40. The reaction of each support is $\frac{1}{2}P$, and the moment is $\frac{1}{2}PX$, and Equation (140) becomes

$$ab = \frac{P}{2E.I} X^2 x.$$

But in this case the greatest deflection is at the middle, and the limits of $\Sigma X^2 x$ are 0 and $\frac{1}{2}l$. Hence, in Fig. 56, let the altitude of the pyramid be $\frac{1}{2}l$, and each side of the base also $\frac{1}{2}l$, and the volume will be

$$\tfrac{1}{2}l \times \tfrac{1}{2}l \times \tfrac{1}{3} \text{ of } \tfrac{1}{2}l = \tfrac{1}{24}l^3$$

$$\therefore AC = \frac{Pl^3}{48E.I}$$

which is the same as Equation (73).

121. CASE IV. LET THE BEAM BE SUPPORTED AT ITS ENDS AND UNIFORMLY LOADED, AS IN FIG. 42.

w being the load on a unit of length, the reaction of each support is $\frac{1}{2}wl$, and its moment at any point of the beam is $\frac{1}{2}wlX$. On the length X there is a load wX, the centre of which is at $\frac{1}{2}X$ from the point considered; hence its moment is $\frac{1}{2}wX^2$, and the total moment is the difference of these moments;

$$\therefore \Sigma PX = \tfrac{1}{2}wlX - \tfrac{1}{2}wX^2,$$

and Equation (140) becomes

$$ab = \frac{w}{2E.I}(lX^2x - X^3x),$$

and the total deflection at the middle is,

$$AC = \frac{w}{2E.I}\left(l\sum_{x=0}^{x=\frac{1}{2}l} X^2x - \sum_{x=0}^{x=\frac{1}{2}l} X^3x\right).$$

The values of the terms within the parentheses have already been found, and by substracting them we have

$$AC = \frac{5}{384}\,\frac{Wl^2}{E.I}.$$

122. REMARKS ABOUT OTHER CASES.—This method, which appears so simple in these cases, unfortunately becomes very complex in many other cases, and in some it is quite powerless. To solve the 9th and 10th cases, pages 124 and 125, necessitates an expression for the inclination of the curve, so that the condition of its being horizontal over the support may be imposed upon the analysis. But the 9th case may be easily solved if we find by any process that the weight which must be suspended at the outer end of the beam to make it horizontal over the support is $\frac{1}{8}Pl$ divided by AC, Fig. 47. For the reaction of the support is $\frac{1}{2}P+P_1$;

$$\therefore PX=P_1(AC+X)-(\tfrac{1}{2}P+P_1)X$$
$$=P_1AC-\tfrac{1}{2}PX$$
$$=\tfrac{1}{8}Pl-\tfrac{1}{2}PX$$
$$ab=\tfrac{1}{8}\frac{PlXx-4PX^2x}{E.I},$$

and the deflection at the centre $=\frac{1}{8}\frac{P}{E.I}(l\Sigma Xx-4\Sigma X^2x)$ taken between the limits 0 and $\frac{1}{2}l$.

The part ΣXx is the area of a triangle whose base and altitude are each $\frac{1}{2}l$, $\therefore \Sigma Xx=\frac{1}{8}l^2$, and ΣX^2x between the limits 0 and $\frac{1}{2}l$, is $\frac{1}{24}l^2 \therefore AC$ (Fig. 55),

$$=\frac{P}{192}\frac{l^2}{E.I}.$$

All these expressions contain I, the value of which remains to be found by the graphical method.

123. MOMENT OF INERTIA OF A RECTANGLE.

Required the moment of inertia of a rectangle about one end as an axis.

Let $ABCD$, Fig. 58, be a rectangle. Make BG perpendicular to and equal AB, and complete the wedge $G-ABCD$.

Let $\overline{\Delta a}=$ the area of a very small surface at E, and $y=AE=EF$, then

$y\overline{\Delta a}=$ the volume of a very small prism EF, and this multiplied by y gives

$y^2\overline{\Delta a}=$the moment of inertia of the elementary area at E, which is also the statical moment of the prism EF, and

$\Sigma y^2\overline{\Delta a}=I=$the moment of inertia of the rectangle $ABCD$.

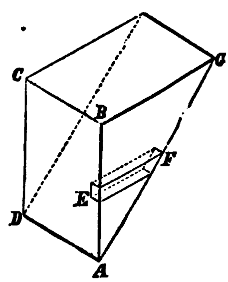

FIG. 58.

Hence the moment of inertia of the rectangle is *represented* by the statical moment of the wedge $G-ABCD$. If

$$AB=d=BG, \text{ and}$$
$$AD=b,$$

then the volume of the wedge is

$$bd\times\tfrac{1}{2}d=\tfrac{1}{2}bd^2$$

and the moment $=\frac{1}{2}bd^2\times\frac{2}{3}d=\frac{1}{3}bd^2$..............................)143)

If the axis of moments passes through the centre of the rectangle, and parallel to one end, we have $BE = GB = \frac{1}{2}d$ in Fig. 59. Hence the moment of inertia of the rectangle =

$$2 \times b \times \tfrac{1}{2}d \times \tfrac{1}{2}d \times \tfrac{2}{3} \text{ of } \tfrac{1}{2}d = \tfrac{1}{12}bd^3$$

which is the same as Equation (51).

124. THE MOMENT OF INERTIA OF A TRIANGLE about an axis parallel to the base and passing through the vertex is, in a similar way, the statical moment of the pyramid $ABCDE$, Fig. 60.

Let $b = CB$ = base of the triangle, and

$d = AB = BD = CE$ = altitude of the triangle and pyramid and sides of the base of the pyramid.

The volume of the pyramid $= \frac{1}{3}bd^2$.

The centre of gravity is $\frac{3}{4}d$ from the apex, consequently the statical moment is $\frac{1}{3}bd^2 \times \frac{3}{4}d = \frac{1}{4}bd^3$.

But in a triangular beam the neutral axis passes through the centre of gravity of the triangle, and it is desirable to find the moment of inertia about an axis which passes through the centre and parallel to the base.

This may be done as in the preceding Article; but it may be more easily

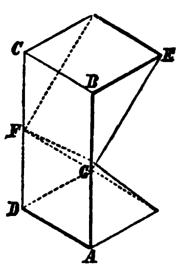

FIG. 59.

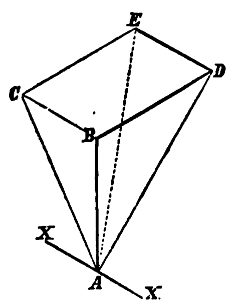

FIG. 60.

done by using the *formula of reduction*, which is as follows:—*The moment of inertia of a figure about an axis passing through its centre equals the moment of inertia about an axis parallel to it, minus the area of the figure multiplied by the square of the distance between the axes.* (See Appendix.)

This gives for the moment of inertia of a triangle about an axis passing through its centre and parallel to the base

$$\tfrac{1}{4}bd^3 - \tfrac{1}{2}bd \times (\tfrac{2}{3}d)^2 = \tfrac{1}{36}bd^3 \ldots\ldots\ldots\ldots(143)$$

125. THE MOMENT OF INERTIA OF A CIRCLE may be represented in the same way, but it is not easy to find the volume of the wedge, or the position of its centre of gravity, except by an analysis which is more tedious than that required to find the moment directly, as was done in Equation (51). But it may be found practically, by those who can only per-form multiplication, as follows :—Make a wedge-shaped piece out of wood, or plaster-of-Paris, or other convenient material, the base of which is the semicircle required, and whose altitude equals the radius of the circle, as shown in Fig. 60*a*; then find its volume by immersing it in a liquid and measuring the amount of water displaced. Then determine the horizontal distance to the centre of gravity of the wedge from the centre of the circle by balancing it on a knife edge, holding the edge of the knife under the base of the wedge, and parallel to the edge, *ab*, of the wedge, keeping the side vertical, and measuring the distance between the edge *ab* and the line of support. Then the statical moment of the wedge, which equals the moment of inertia of the semicircle, is the product of the volume multiplied by the horizontal distance of the centre from the edge, and twice this amount is the moment of inertia of the whole circle. Its value for the whole circle, or for both wedges, is $\frac{1}{4}\pi r^4$.

There are, however, many methods of calculating the moment of inertia of a circle without using the Calculus. The following appears as simple as any of the known methods :—

The moment of inertia of a circle is the same about all its diameters. Hence the moment about X in the figure, *plus* the moment about Y, equals *twice* the moment about X. The distance to any point A is ρ, and equals $\sqrt{x^2+y^2}$; or $\rho^2 = x^2 + y^2$; and if $\overline{\Delta a}$ be an elementary area, as before, we have

$$2\Sigma\overline{\Delta a}\, x^2 = \Sigma\overline{\Delta a}\, x^2 + \Sigma\overline{\Delta a} y^2 = \Sigma\overline{\Delta a}\, \rho^2,$$

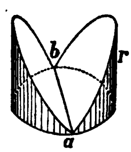

FIG. 60*a*

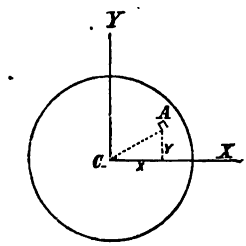

FIG. 60*b*.

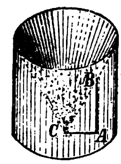

FIG. 60*c*.

the latter of which is called the polar moment of inertia, in reference to an axis perpendicular to the plane of the circle, and passing through its centre C. To find the value of $\Sigma\overline{\Delta a}\, \rho^2$, take a triangle whose base and altitude are each equal to r, the radius of the circle, and revolve it about the axis through C, and construct an infinitely small prism on the element $\overline{\Delta a}$ as a base.

We have $\rho = CA = AB$, Fig. 60*c*.
$\overline{\Delta a}\, \rho =$ volume of the small prism AB.
$\overline{\Delta a}\, \rho\, CA = \overline{\Delta a}\, \rho^2 =$ the statical moment of AB,
which expression is of the form of the quantity sought.

Hence $\Sigma\overline{\Delta a}\, \rho^2$ is the product of the volume of the solid generated by the triangle, multiplied by the abscissa of its centre of gravity from C. The solid is what remains of a cylinder after a cone has been taken out of it, the base of the cone being the upper base of the cylinder, and the apex of which is at the centre of the base of the cylinder. Hence the volume of the solid is the volume of the cylinder, *less* the volume of the cone;

$$\text{or } \pi r^2 \times r - \pi r^2 \times \tfrac{1}{3}r = \tfrac{2}{3}\pi r^3.$$

If now the solid be divided into an infinite number of pieces, by planes which pass through its axis, each small solid will be a pyramid, having its vertex at C, and the abscissa to the centre of gravity of each will be $\frac{3}{4}r$ from C. Hence we finally have

$$\Sigma \overline{\Delta a}\, \rho^2 = \tfrac{2}{3}\pi r^3 \times \tfrac{3}{4}r = \tfrac{1}{2}\pi r^4,$$

which equals $2\Sigma \overline{\Delta a}\, x^2$.

$$\therefore \Sigma \overline{\Delta a}\, x^2 = \tfrac{1}{4}\pi r^4 \quad \text{.............................(144)}$$

126. MOMENT OF INERTIA OF OTHER SURFACES.—The general method indicated in the preceding articles is applicable to surfaces of any character, and with careful manipulation approximations may be made which will be very nearly correct, and, as we have seen above, in some cases exact formulas may be found.

127. VIBRATIONS OF BEAMS.—If a load be placed suddenly upon a beam, and be left to the action of the elastic forces, it will vibrate. Or if a load is upon the beam and the deflection be increased or decreased by an external force, and then left to the action of the elastic forces, it will vibrate the same as before. Take the case shown in Fig. 36, and suppose that the weight is applied suddenly.

Let z be the variable deflection; then from Eq. (57) we find that the pressure, P', which will produce this deflection is

$$P' = \frac{3EI}{l^3} z,$$

and hence the pressure which is still available for producing the maximum deflection is

$$\frac{3EI}{l^3}(\Delta - z)$$

From Mechanics we have $\frac{d^2z}{dt^2}$ = the acceleration and $\frac{P}{g}\frac{d^2z}{dt^2}$ = the moving pressure $= \frac{3EI}{l^3}(\Delta - z)$.

Integrating once gives

$$\frac{P}{g}\frac{dz^2}{dt^2} = \frac{3EI}{l^3}(2\Delta z - z^2)$$

$$\therefore dt = l\sqrt{\frac{Pl}{3gEI}}\,\frac{dz}{\sqrt{2\Delta z - z^2}}$$

$$\therefore t = l\sqrt{\frac{Pl}{3gEI}}\,\text{ver sin}^{-1}\frac{s}{\Delta}$$

For $s = \Delta$, we have

$$t = \tfrac{1}{2}\pi\sqrt{\frac{Pl^3}{3gEI}} = \tfrac{1}{2}\pi\sqrt{\frac{\Delta}{g}};$$

hence they are isochronous. The weight of the beam has been neglected. We would find a similar expression if the beam were uniformly loaded, or if supported at its ends.

CHAPTER VI.

TRANSVERSE STRENGTH.

128. STRENGTH OF RECTANGULAR BEAMS.—The theories which have been advanced from time to time to explain the mechanical action of the fibres, have been already given in Chapter IV.

First, consider the common theory, according to which the neutral axis passes through the centre of gravity of the transverse sections, and the strain upon the fibres is directly proportional to their distance from the neutral axis.

Continuing the use of the geometrical method, let Fig. 61 represent a rectangular beam which is strained by a force P applied at any point. Let de be on the neutral axis, and ab represent the strain upon the lowest fibre. Pass a plane, de–cb, and the wedge so cut off *represents* the strains on the lower side, and the similar wedge on the other side *represents* the strains on the upper side.

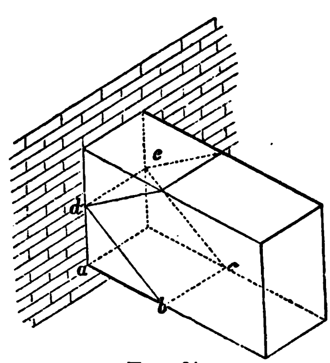

FIG. 61.

Let R = the ultimate strain upon a unit of fibres most remote from the neutral axis on the side which first ruptures, on the hypothesis that all the fibres of the unit are equally strained, and b = the breadth and d = the depth of the beam.

Let $ab = R$; then, the total resistance to compression $= \frac{1}{2}Rb \times \frac{1}{2}d = \frac{1}{4}Rbd$, = the volume of the lower wedge; and the *moment of resistance* is this value multiplied by the ordinate to the centre of gravity of the wedge from de, which is $\frac{2}{3}$ of $\frac{1}{2}d = \frac{1}{3}d$; consequently the *moment* is

$$\tfrac{1}{12}Rbd^2;$$

and as the moment of resistance to tension is the same, *the total moment of resistance is*

$$\tfrac{1}{6}Rbd^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (145)$$

which equals the moment of the applied or bending forces.

If the beam be fixed at one end and loaded by a weight, P, at the free end, we have for the *dangerous section*, or that most liable to break,

$$Pl = \tfrac{1}{6}Rbd^2.$$

In rectangular beams the dangerous section will be where the sum of the moments of stresses is greatest, the maximum values of which for a few cases are given in a table on page 128. Using those values, and placing them equal to $\tfrac{1}{6}Rbd^2$, and we have for solid rectangular beams at the dangerous section, the following formulas:—

FOR A BEAM FIXED AT ONE END AND A LOAD, P, AT THE FREE END;

$$Pl = \tfrac{1}{6}Rbd^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (146)$$

AND FOR AN UNIFORM LOAD;

$$\tfrac{1}{2}Wl = \tfrac{1}{6}Rbd^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (147)$$

FOR A BEAM SUPPORTED AT ITS ENDS AND A LOAD, P, AT THE MIDDLE;

$$\tfrac{1}{4}Pl = \tfrac{1}{6}Rbd^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (148)$$

AND FOR AN UNIFORM LOAD;

$$\tfrac{1}{8}Wl = \tfrac{1}{6}Rbd^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (149)$$

AND FOR A LOAD AT THE MIDDLE, AND ALSO AN UNIFORM LOAD;

$$\tfrac{1}{8}(2P + W)l = \tfrac{1}{6}Rbd^2 \ldots\ldots\ldots\ldots\ldots (150)$$

FOR A BEAM FIXED AT BOTH ENDS AND A LOAD, P, AT THE MIDDLE;

$$\tfrac{1}{8}Pl = \tfrac{1}{6}Rbd^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (151)$$

AND FOR AN UNIFORM LOAD, END SECTION;

$$\tfrac{1}{12}Wl = \tfrac{1}{6}Rbd^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (152)$$

MIDDLE SECTION;

$$\tfrac{1}{24}Wl = \tfrac{1}{6}Rbd^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (153)$$

These expressions show that in solid rectangular beams the strength varies as the breadth and square of the depth, and hence breadth should be sacrificed for depth. In all the cases, except for a beam fixed at the ends, it appears that a beam will support twice as much if the load be uniformly distributed over the whole length as if it be concentrated at the middle of the length. The case in which a beam is fixed at both ends and loaded at the middle has given rise to considerable discussion, for it is found by experiment that a beam whose ends are fixed in walls of masonry will not sustain as much as is indicated by the formula, and also that it requires considerably more load to break it at the ends than at the middle, but the analysis shows that it is equally liable to break at the ends or at the middle. But it should be observed that there is considerable difference between the condition of mathematical fixedness, in which case the beam is horizontal over the supports, and that of embedding a beam in a wall. For in the latter case the deflection will extend some distance into the wall.

Mr. Barlow concludes from his experiments that Equation (151) should be

$$\tfrac{1}{6}Pl = \tfrac{1}{6}Rbd^2 \qquad (154)$$

and this relation is doubtless more nearly realized in practice than the ideal one given above. In either case, it appears that writers and experimenters have entirely overlooked the effect due to the change of position of the neutral axis, which must take place. It has been assumed that the neutral axis coincides with the axis of the beam, and that its length remains unchanged during flexure; but if the ends of the beam are fixed, the axis must be elongated by flexure, or else approach much nearer the concave than the convex side, or both take place at the same time, in which case the *moment of resistance* will not be $\tfrac{1}{6}Rbd^2$. The phenomena are of too complex a character to admit of a thorough and exact analysis, and it is probably safer to accept the results of Mr. Barlow in practice than depend upon theoretical results.

129. MODULUS OF RUPTURE.—When a beam is supported at its ends, and loaded uniformly over its whole length, and also loaded at the middle, we find from Equation (150)

$$R = \frac{\frac{3}{4}(2P + W)}{bd^2} l \ldots\ldots\ldots\ldots\ldots\ldots(155)$$

in which W may be the weight of the beam. Beams of known dimensions, thus supported, have been broken by weights placed at the middle of the length, and the corresponding value of R has been found for various materials, the results of which have been entered in the table in Appendix III. This is called the MODULUS OF RUPTURE, and is defined to be *the strain at the instant of rupture upon a square inch of fibres most remote from the neutral axis on the side which first ruptures.* It would seem from this definition that R should equal either the tenacity or crushing resistance of the material, depending upon whether it broke by crushing or tearing, but an examination of the table shows the paradoxical result that it never equals either, but is always greater than the smaller and less than the greater.

The tabulated values of R being found from experiments upon solid rectangular beams, they are especially applicable to all beams of that form, and they answer for all others that do not depart largely from that form; but if they depart largely from that form, as in the case of the **I** (double T) section, or hollow beams, or other irregular forms, the formulas will give results somewhat in excess of the true strength; and in such cases Barlow's theory gives results more nearly correct.

But if, instead of R, we use T or C, whichever is smaller, in the formulas which we have deduced, and suppose that the neutral axis remains at the centre of the beam, *we shall always be on the safe side;* but there would often be an excess of strength, as, for instance, in the case of cast-iron the actual strength of the beam would be about twice as strong as that found by such a computation.

The difficulty is avoided, practically, by using such a small fractional part of R as that it will be considered perfectly safe. This fraction is called the *coefficient of safety.* The values commonly used for beams are the same as for bars, and are given in Article 43.

Experiments should be made upon the material to be used in a structure, in order to determine its strength; but in the absence of such experiments the following mean values of R are used:—

850 to 1,200 lbs. for wood,
10,000 to 15,000 lbs. for wrought-iron, and
6,000 to 8,000 lbs. for cast-iron.

130. PRACTICAL FORMULAS.

If $R = 1{,}000$ for wood, and
12,000 for wrought-iron,

we have for a rectangular beam, supported at its ends and loaded at the middle of its length,

$$P = \frac{666\, bd^2}{l} \text{ for wooden beams; and}$$

$$P = \frac{8000\, bd^2}{l} \text{ for wrought-iron beams.}$$

The length of the beam, and the load it is to sustain, are generally known quantities, and the breadth and depth are required; but it is necessary to assume one of the latter, or assign a relation between them. For instance, if the depth be n times the breadth, the preceding formulas give

$$b = \sqrt[3]{\frac{Pl}{666n^2}}; \text{ and } d = \sqrt[3]{\frac{Pln}{666}} \text{ for wood} \ldots\ldots (156)$$

$$\text{and } b = \sqrt[3]{\frac{Pl}{8000n^2}}; \text{ and } d = \sqrt[3]{\frac{Pln}{8000}} \text{ for wrought-iron; } (157)$$

131. THE RELATIVE STRENGTH OF A BEAM under the various conditions that it is supported or held is as the moment of the applied forces; hence, all the cases which have been considered may, *relatively*, be reduced to *one*, by finding how much a beam will carry which is fixed at one end and loaded at the free end, Equation (146), and multiplying the results by the following factors:—

	FACTORS.
Beam fixed at one end and loaded at the other	1
" " " " uniformly loaded	2
Beam supported at its ends and loaded at the middle	4
" " " " uniformly loaded	8
Beam fixed at one end and supported at the other, and uniformly loaded	8
Beam fixed at both ends and loaded at the middle	8
" " " " uniformly loaded	12

If it is required to know the breadth of a beam which will sustain a given load, find b, from Equation (146); and for a beam in any other condition, divide by the factors given above for the corresponding case.

If the depth is required, find d, from Equation (146), and divide the result for the particular case desired by the square root of the above factors.

132. EXAMPLES.

1. A beam, whose depth is 8 inches, and length 8 feet, is supported at its ends, and required to sustain 500 pounds per foot of its length; required its breadth so that it will have a factor of safety of $\frac{1}{10}$, R being 14,000 pounds.

From Equation (146) we have,

$$b = \frac{6Pl}{Rd^2} = \frac{6 \times 500 \times 8 \times 8 \times 12}{1400 \times 8^2} = 25\tfrac{5}{7} \text{ inches};$$

and by examining the above table of factors we see that this must be divided by 8; $\therefore$ Ans. $3\frac{3}{14}$ inches.

2. If $l = 10$ feet, P at the middle $= 2{,}000$ lbs., $b = 4$ inches, $R = 1{,}000$ lbs., required d. Ans. 9.48 inches.

3. If a beam, whose length is 8 feet, breadth is 3 inches, and depth 6 inches, is supported at its ends, and is broken by a weight of 10,000 pounds placed at the middle, and the weight of a cubic foot of the beam is 50 pounds; required the value of R. Use Equation (150).

4. If $R = 80{,}000$ lbs., $l = 12$ feet, $b = 2$ inches, $d = 5$ inches, how much will the beam sustain if supported at its ends and loaded uniformly over its whole length, coefficient of safety $\frac{1}{4}$? Ans. $W = 9{,}259$ lbs.

5. A wooden beam, whose length is 12 feet, is supported at its ends; required its breadth and depth so that it shall sustain one ton, uniformly distri-

buted over its whole length. Let $R = 15,000$ lbs., coefficient of safety $\frac{1}{10}$, and depth $= 4$ times the breadth. Ans. $b = 2.08$ inches
$d = 8.32$ inches.

6. A beam is 2 inches wide and 8 inches deep, how much more will it sustain with its broad side vertical, than with it horizontal?

7. A wrought-iron beam 12 feet long, 2 inches wide, 4 inches deep, is supported at its ends. The material weighs $\frac{1}{4}$ lb. per cubic inch; how much load will it sustain uniformly distributed over its whole length, $R = 54,000$ lbs.?
Ans. Without the weight of the beam, 15,712 lbs.

8. A beam is fixed at one end; $l = 20$ feet, $b = 1\frac{1}{2}$ inch, $R = 40,000$ lbs.; weight of a cubic inch of the beam $\frac{1}{4}$ lb. Required the depth that it may sustain its own weight and 500 lbs. at the free end. Ans. 4.05 inches.

9. The breadth of a beam is 3 inches, depth 8 inches, weight of a cubic foot of the beam 50 pounds, $R = 12,000$; required the length so that the beam shall break from its own weight when supported at its ends.
Ans. $l = 175.27$ feet.

133. RELATION BETWEEN STRAIN AND DEFLECTION.—When the strain is within the elastic limit we may easily find the greatest strain on the fibres corresponding to a given deflection. For instance, take a rectangular beam, supported at its ends and loaded at the middle of its length, and we have from Equation (148)

$$P = \frac{2}{3}\frac{Rbd^2}{l}$$

and from Equations (73) and (51)

$\Delta = \frac{1}{4}\frac{Pl^3}{Ebd^3}$, which becomes, by substituting P from the preceding,

$$\Delta = \frac{1}{6}\frac{Rl^2}{Ed}$$

$$\therefore R = \frac{6Ed}{l^2}\Delta \quad \text{(158)}$$

Examples.—1. If $l = 6$ feet, $b = 1\frac{1}{2}$ inch, $d = 4$ inches, coefficient of elasticity $= 25,000,000$ lbs. is supported at its ends and loaded at the middle so as to produce a deflection at the middle of $\Delta = \frac{3}{4}$ inch; required the greatest strain on the fibres. Also required the load.

2. On the same beam, if the greatest strain is $R = 12,000$ lbs., required the greatest deflection.

3. If the beam is uniformly loaded, required the relation between the greatest strain and the greatest deflection.

134. HOLLOW RECTANGULAR BEAMS.—If a rectangular beam has a rectangular hollow, both symmetrically placed in reference to the neutral axis, as in Fig. 62, we may find its strength by deducting from the strength of a solid rectangular beam the strength of a solid beam of the same size as the hollow. But in this case, when the beam ruptures at b, the strain at b' will be less than R. As the strains increase directly as the distance of the fibres from the neutral axis, we have, if d and d' are the depth of the outside and hollow parts respectively,

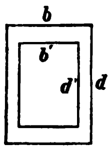

FIG. 62.

$$\tfrac{1}{2}d : \tfrac{1}{2}d' :: R : \text{strain at } b' = R\frac{d'}{d}.$$

If b' = the breadth of the hollow, the stress on that part, if it were solid, would be, according to Equation (145),

$$\tfrac{1}{6}\left(R\frac{d'}{d}\right)b'd'^2 = \tfrac{1}{6}R\frac{b'd'^3}{d},$$

which, taken from Equation (145), gives for the resistance of a hollow rectangular beam,

$$\tfrac{1}{6}R\,\frac{bd^3 - b'd'^3}{d} \qquad (159)$$

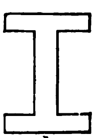

FIG. 63.

If the hollow be on the outside, as in Fig. 63, forming an H section, the result is the same.

135. IF THE UPPER AND LOWER FLANGES ARE UNEQUAL it forms a double T, as in Fig. 64. Let the notation be as in the figure, and also d_1 equal the distance from the neutral axis to the upper element, and x the distance from the neutral axis to the lower element.

To find the position of the neutral axis, make the statical moments of the surface above it equal to those below it. This gives

$$d' b' (d_1 - \tfrac{1}{2} d') + \tfrac{1}{2} b''' (d_1 - d')^2 = d'' b'' (x - \tfrac{1}{2} d'') + \tfrac{1}{2} b''' (x - d'')^2 \quad \text{.......... (160)}$$

We also have $d_1 = d - x = d' + d'' + d''' - x$.. (161)

These equations will give x and d_1.

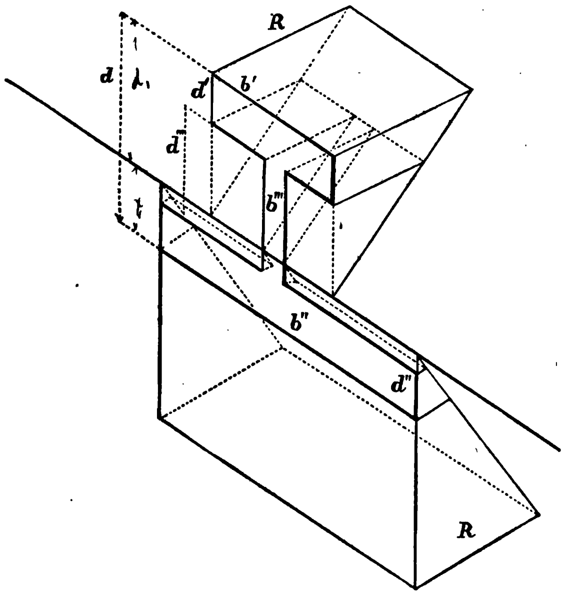

FIG. 64.

Constructing the wedges as before, and the resistance to compression is represented by the wedge whose base is $b' d_1$ and altitude R, *minus* the wedge whose base is $(b' - b''') (d_1 - d')$ and altitude $\frac{d_1 - d'}{d_1} R$. Hence the resistance to compression is

$$\tfrac{1}{2} R b' d_1 - \tfrac{1}{2} \frac{d_1 - d'}{d_1} R (b' - b''') (d_1 - d')$$

The centre of gravity is at $\frac{2}{3}$ the altitude, or $\frac{2}{3} d_1$ for the former wedge, and $\frac{2}{3}(d_1 - d')$ for the latter, and if the volumes be multiplied by these quantities respectively, it will give for *the moment of resistance to compression*

$$\tfrac{1}{3} R b' d_1^2 - \tfrac{1}{3} \frac{R}{d_1} (b' - b''') (d_1 - d')^3$$

Next consider the resistance to tension. Since the strains on the elements are proportional to their distances from the neutral axis, therefore

$d_1 : x :: R :$ strain at the lower side of the section $= \frac{R}{d_1} x$,

and similarly,

$d_1 : (x - d'') :: R :$ strain at the opposite side of the lower flange $= \frac{R}{d_1} (x - d'')$.

Hence the tensive strains will be represented by a wedge whose base is $b''x$ and altitude $\frac{R}{d_1} x$, *minus* a wedge whose base is $(b'' - b''') (x - d'')$ and altitude $\frac{R}{d_1} (x - d'')$. Hence the moment of resistance is

$$\tfrac{1}{3} \frac{R}{d_1} b'' x^3 - \tfrac{1}{3} \frac{R}{d_1} (b'' - b''') (x - d'')^3$$

The total moment of resistance is the sum of the two moments, or

$$\tfrac{1}{3} \frac{R}{d_1} \Big[b' d_1^3 - (b' - b''') (d_1 - d')^3 + b'' x^3 - (b'' - b''') (x - d'')^3 \Big] \quad \ldots\ldots(162)$$

For a single T make b'' and $d'' = 0$ in the above expression. The method which has here been applied to rectangular beams may be applied to beams of any form; but it often requires a knowledge of higher mathematics to find the volume of the wedge, and the position of its centre of gravity; or resort must be had to ingenious methods in connection with actual wedges of similar dimensions.

136. TRUE VALUE OF d_1 AND AN EXAMPLE.—In this and similar expressions

$d_1 =$ *the distance from the neutral axis to the fibre most remote from it* ON THE SIDE WHICH FIRST RUPTURES.

d_1 is usually taken as the distance to the most remote fibre, without considering whether rupture will take place on that side or not; but this oversight may lead to large errors.

For example, let the dimensions of a cast-iron double T-beam be as in Fig. 65, and 228 inches between the supports. Required the load at the middle necessary to break it.

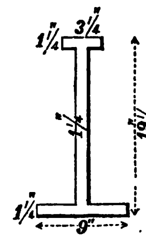

FIG. 65.

The position of the neutral axis is found from Equations (160) and (161) to be 7.96 inches from the lower side, and 11.54 inches from the upper. As cast-iron will resist from four to six times as much to compression as to tension—this beam will rupture on the lower side first; hence d_1 in the equation = 7.96 inches. As the value of R is not known, take a mean value = 36,000 lbs. The moment of the rupturing force—neglecting the weight of the beam—is $\frac{1}{4} Pl$, which placed equal to Expression (162) and reduced gives

$$P = \frac{4}{228} \times \frac{36,000}{7.96} \times 1,672 = 132,000 \text{ lbs.} = 58.9 \text{ tons gross.}$$

Had we used $d_1 = 11.54$, it would have given $P = 40.0$ tons. Such beams actually broke with from 50 to 54 tons; or, including the weight of the beam, with a mean value of $52\frac{1}{2}$ tons.

By reversing the problem, and using $52\frac{1}{2}$ tons for P, we find that R is a little more than 32,000 pounds. Had this value of R been used in the first solution, and d_1 made equal 11.54, it would have given for P a little more than 36 tons, which would be the strength if the beam were inverted. If the upper flange were smaller or the lower larger, the discrepancy would have been greater.

The strain upon a fibre in the upper surface is to the strain upon one in the lower surface as d_1 to x; hence, if the material resists more to compression than to tension (as cast-iron), it should be so placed that the small flange shall resist the former, and the large one the latter. If a cast-iron beam be supported at its ends, the smaller flange should be uppermost, and as it resists from four to six times as much compression as tension, the neutral axis should be from four to six times as far from the upper surface as from the lower, for economy. Using the same notation as in Fig. 64, and we have,

$$\frac{d_1}{x} = \frac{\text{greatest compressive strain}}{\text{greatest tensive strain}},$$

and for economy we should have,

$$\frac{d_1}{x} = \frac{\text{ultimate compressive strength}}{\text{ultimate tensile strength}}.$$

The ultimate resistance of wrought-iron is greater for tension than for compresaion; hence, if a wrought-iron beam is supported at its ends, the heavier flange should be uppermost.

The proper thickness of the vertical web can bc determined only by experiment, and this has been done, in a measure, by Baron von Weber, in his experiments ou permanent way.

137. EXPERIMENTS OF BARON VON WEBER for determining the thickness required for the central web of rails.

Baron von Weber desired to ascertain what was the *minimum* thickness which could be given to the web of a rail, in order that the latter might still possess a greater power of resistancc to lateral forces than the fastenings by which it was

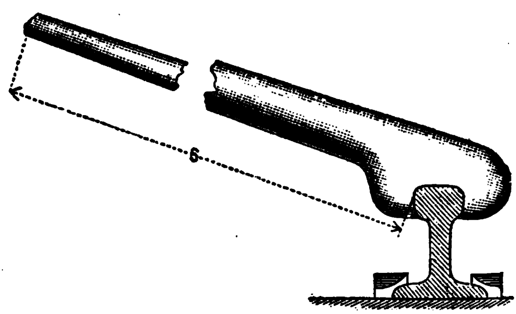

FIG. 65*a*.

secured to the sleepers. For this purpose a piece of rail 6 feet in length, rolled, of the best iron at the Laurahutte, in Silesia, was supported at distances of 35.43 in., and loaded nearly to the limit of elasticity (which had been determined previously by experiments on other pieces of the same rail), and the deflections were then measured with great care by an instrument capable of registering $\frac{1}{1000}$ in. with accuracy. This having been done, the web of the piece of rail was planed down, and each time that the thickness had been reduced 3 millimetres the vertical deflection of the rail under the above load was again tested, and the rail was subjected to the following rough but practical experiments. The piece of rail was fastened to twice as many fir sleepers by double the number of spikes which would be employed in practice, and a lateral pressure was then applied to the head of the rail by means of a lifting-jack, until the rail began to cant and the spikes were drawn. The same thing was then done by a sudden pull, the apparatus used being a long lever fastened to the top of the rail, as shown in Fig. 65*a*. The lifting-jack and the lever were applied to the ends of the rail, and the web of the latter had, in each case,

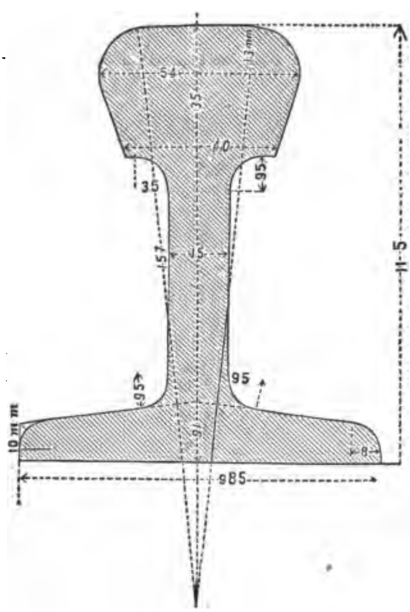

FIG. 65*b*.

to resist the whole strain required for drawing out the spikes. The results of the experiments made to ascertain the resistance of the rail to vertical flexure with different thicknesses of web, and under a load of 5,000 lbs., were as follows:—

	Thickness of web. In.	Vertical deflection. In.
15 millimetres =	0.59	0.016
12 "	0.47	0.016
9 "	0.35	0.019
6 "	0.24	0.0194
3 "	0.12	0.022

These results showed ample stiffness, even when the web was reduced in thickness to 0.12 in. To determine the power of resistance of the rail to lateral flexure, an impression of the section was taken in lead each time that the spikes were drawn.

The forces applied in these experiments were very far greater than those occurring in practice, yet it was found that with the web 12, 9, and even 6 millimetres thick, no distortion took place, and only when the thickness of the web was reduced to 3 millimetres (0.12 in.) was a slight permanent lateral deflection of the

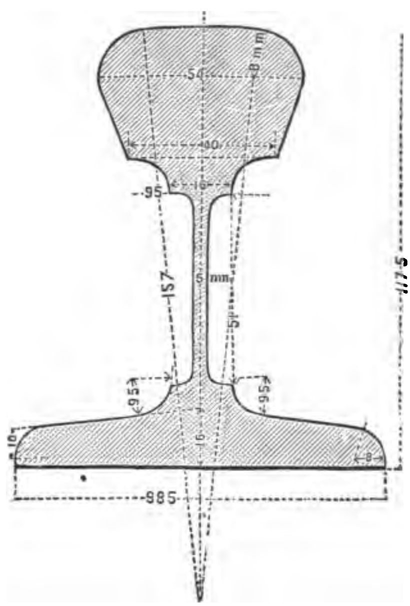

FIG. 65c.

head caused just as the spikes gave way. The section shown in Fig. 65b had then been reduced to that shown in Fig. 65c.

Next, a rail, with the web reduced to 3 mill. (0.12 in.) in thickness, was placed in the line leading to a turn-table on the Western Railway of Saxony, where it has remained until the present time, 1870, receiving the shocks due to engines passing to and from the turn-table more than one hundred times daily.

It follows from these experiments that the least thickness ever given to the webs of rails in practice is more than sufficient, and that if it were possible to roll webs $\frac{1}{4}$ in. thick, such webs would be amply tirong, if it were not that there would be a chance of their being torn at the points where they are traversed by the fish-plate bolts. Baron von Weber concludes that webs $\frac{3}{8}$ in. or $\frac{1}{2}$ in. thick are amply strong enough for rails of any ordinary height, and that, in fact, the webs should be made as thin as the process of rolling and as the provision of sufficient bearing for the fish-plate bolts will permit.

138. ANOTHER GRAPHICAL METHOD.—If elementary processes are to be used for determining the strength, the following method possesses many advantages over the former.

Since the strains vary directly as their distance from the neutral axis, the triangle ABC (Fig. 66), in the rectangle $BCDE$, represents the compressive strains if each element of the shaded part has a strain equal to R; and its moment is R times the area multiplied by the distance of the centre of gravity of the triangle from the neutral axis; or,

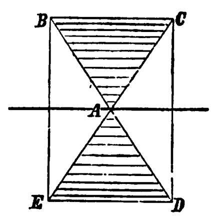

FIG. 66.

$$R \times (b \times \tfrac{1}{2} \text{ of } \tfrac{1}{2}d) \times \tfrac{2}{3} \text{ of } \tfrac{1}{2}d = \tfrac{1}{12}Rbd^2,$$

and the moment of tensile resistance is the same, hence the total moment is double this, or $\frac{1}{6}Rbd^2$, as found by the preceding process.

139. IF A SQUARE BEAM HAVE ONE OF ITS DIAGONALS VERTICAL (Fig. 67), the neutral axis will coincide with the other diagonal. Take any element, as ab, and project it on a line cd, which passes through A and is parallel to BC, and draw the lines Oc and Od, and note the points f and g where they intersect the line ab. If the element were at cd, the strain upon it would be R, multiplied by the area of cd, or simply $R.cd$; but because the strains are directly proportional to the distances of the elements from the neutral axis, the strain on ab is $R.fg$. Proceed in this way with all the elements and construct the shaded figure. The strains on the upper part of the figure

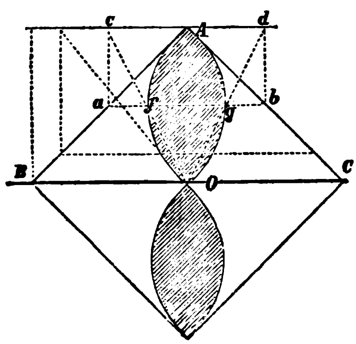

FIG. 67

ABC, which begin with zero at BC, and increase gradually to R, at A, will be equivalent to the strains on the shaded figure AO, if the strain is equal to R on each unit of its surface. Hence the total strain on each half is the area of the shaded part AO, multiplied by R, and the moment of the strain of each part is this product multiplied by the distance of the centre of the shaded part from the axis BC.

By similar triangles we have

$$Aa : ab :: AB : BC, \text{ and}$$

$$cd = ab : fg :: AO : x :: AB : Ba \text{ or } AB - Aa;$$

x being the distance of fg from O.

From these eliminate ab, and find

$$af = \tfrac{1}{2}(ab - fg) = \tfrac{1}{2}\frac{BC}{(AB)^2}(Aa)^2$$

hence the curve which bounds the shaded figure is a parabola which is tangent to AB, and whose axis is parallel to BC.

Let $d =$ one side of the square, then

$\sqrt{2}d = BC$,
$\frac{1}{2}\sqrt{2}d = AO$, and
$\frac{1}{4}\sqrt{2}d =$ the widest part of the shaded figure.

The area of a parabola is two-thirds the area of a circumscribed rectangle.

Hence the area of AO is

$$\tfrac{2}{3} \times \tfrac{1}{2}\sqrt{2}d \times \tfrac{1}{4}\sqrt{2}d = \tfrac{1}{6}d^2,$$

and the moment is

$$\tfrac{1}{6}d^2 \times \tfrac{1}{4}\sqrt{2}d = \frac{d^3}{12\sqrt{2}},$$

and the moment of both sides, multiplied by R, is

$$R\frac{d^3}{6\sqrt{2}} \dots\dots\dots\dots (163)$$

If $b = d$ in Equation (145) and the result compared with the above, we find:—

The strength of a square beam with its side vertical : strength of the same beam with one of its diagonals vertical :: $\sqrt{2} : 1$ or as $7 : 5$ nearly.

So that increased depth merely is not a sufficient guarantee of increased strength. The reason why the strength is diminished when the diagonal is vertical, is because there is a very small area at the vertex where the strain is greatest, but when a side is horizontal the whole width resists the maximum strain.

140. IRREGULAR SECTIONS.—This method is applicable to irregular sections, as shown by the following example.

Let Fig. 68 be a cross section of a beam. In a practical case it may be well to make an exact pattern of the cross section, of stiff paper or of a thin board of uniform thickness. To find the position of the neutral axis, draw a line on the pattern which shall be perpendicular to the direction of the forces which act upon the beam, that is, if the forces are vertical the line will be horizontal. In a form like Fig. 68, this line will naturally be parallel te the base of the figure. Then balance the pattern on a knife-edge, keeping the base of the figure (or the *line* previously drawn) parallel to the knife-edge, and when it is balanced the line of support will be the neutral axis. Proceed to construct the shaded part as shown in the figure, by pro-

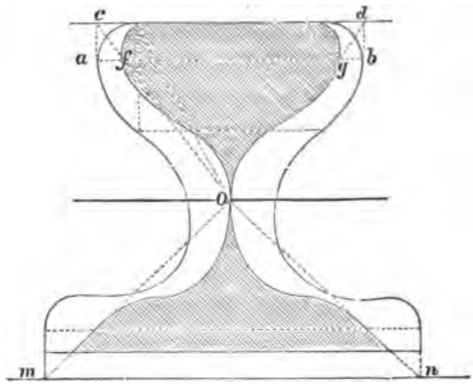

FIG. 68.

jecting any element, as *ab* on the line *cd*, and drawing *cO* and *dO*, and noting the intersections *f* and *g*, the same as in Fig. 67. The elements on the lower side must be projected on a line *mn*, which is at the same distance from the neutral axis as the most remote element on the upper side. The area of the shaded part above the neutral axis should equal that below, because the resistance to extension equals that for compression. The area of the shaded part may be found approximately by dividing it into small rectangles of known size, and adding together the full rectangles and estimating the sum of the fractional parts. Or, the shaded part may be cut out and carefully weighed and balanced by a rectangle of the same material, after which the sides of the rectangle may be carefully measured and contents computed. The area of the rectangle would evidently equal the area of the irregular figure.

The ordinate to the centre of gravity of each part may be determined by cutting out the shaded parts and balancing each of them separately on a knife-edge, as before explained, keeping the knife-edge parallel to the neutral axis. The distance between the line of support and the neutral axis will be the ordinate to the centre of gravity. *The moment of resistance is then found by multiplying the area of each shaded part by* *the distance of its centre of gravity from the neutral axis, and multiplying the sum of the products by R.*

These mechanical methods may be managed by persons who have only a very limited knowledge of mathematics, and if skilfully and carefully done will give satisfactory results. It does not, however, furnish such an *uniform*, direct, and *exact* mode of solution as the analytical method which is hereafter explained.

141. FORMULA OF STRENGTH ACCORDING TO BARLOW'S THEORY.—Either of the above methods may be used. One part of the *expression* for the strength is of the same *form* as that found by the common theory; but instead of R we must use T, or C—the former if it ruptures by tension, the latter if by crushing. The other resistance, ϕ, for solid beams is evenly distributed over the surface. For example, take a rectangular beam, Fig. 61, and the resistance to *longitudinal shearing* on the upper side is $\phi\, b \times \frac{1}{2}d = \frac{1}{2}\,\phi\, bd$, and its moment is $\frac{2}{3}\,\phi\, bd \times \frac{1}{2}$ of $\frac{1}{2}d = \frac{1}{8}\,\phi\, bd^2$, and for both sides, $\frac{1}{4}\,\phi\, bd^2$. Hence, according to Barlow's theory, the expression for the strength of a rectangular beam is

$$[\tfrac{1}{4}\,\phi + \tfrac{1}{6}T]\, bd^2 \text{ for cast-iron, and}$$

$$[\tfrac{1}{4}\,\phi + \tfrac{1}{6}C]\, bd^2 \text{ for wrought-iron and wood} \quad\quad (164)$$

If the beam is supported at its ends and loaded at the middle, we have

$$\tfrac{1}{4}Pl = [\tfrac{1}{4}\,\phi + \tfrac{1}{6}T]\, bd^2 \text{ for cast-iron} \quad\quad (165)$$

The volume which represents the resistance due to ϕ is always a prism, having for its base the surface of the figure and ϕ, or some fraction of ϕ, for its altitude. If the second method of illustration be used, it will take two figures to fully illustrate the strains. For instance, if the section be as in Fig. 68, the moment of the shaded part will be multiplied by T or C, as the case may be. To find the remaining part of the moment, find the area of each part of the transverse section, also the distance of the centre of gravity of each part from the neutral axis. Then, *to find the moment of resistance due to longitudinal shearing, multiply the area of each part by the distance of its centre of gravity from the neutral axis, add the products and multiply the sum by ϕ.* This is true for solid sections; but for hollow beams, T and H sections, where there is an abrupt angular change from the flange to the vertical part of the beam, the factor ϕ requires a modification.

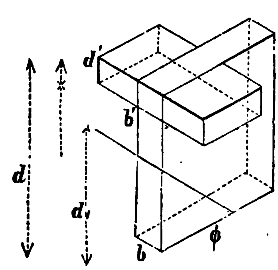

FIG. 69.

For instance, take the simple case of a single T, Fig. 69, in which the breadth of the T is b' and its depth d', and the other notation as in the figure.

The resistance of the upper part is represented by the prism whose base is

bx, and whose altitude is ϕ plus the prism whose base is d' $(b'-b)$, *and whose altitude is* $\frac{d'}{x}\phi$. The resistance of the lower part is $\phi\, bd_1$. The total moment of this resistance is—

$$\phi\, bx.\tfrac{1}{2}x + d'(b'-b) \times \frac{d'}{x}\phi\,(x - \tfrac{1}{2}d') + \phi bd_1 \times \tfrac{1}{2}d_1.$$

To this add the moment of resistance for direct extension and compression, the expression for which is of the same form as for common theory, and we have for the total moment:—

$$\tfrac{1}{2}\phi bx^2 + \frac{d'^2}{x}\phi(b'-b)\,(x-\tfrac{1}{2}d') + \tfrac{1}{2}\phi\, bd_1^2 + \frac{T}{3d_1}\,[bd_1^3 + b'x^3 - (b'-b)\,(x-d')^3] \quad \text{.......(166)}$$

From numerous experiments made upon cast-iron beams having a variety of cross sections, Barlow found that ϕ varied nearly as T, that practically it was a fraction of T, the *mean* value of which was $0.9T$.

For wrought-iron he found $\phi = 0.53T$
$= 0.6C$ nearly.

Peter Barlow, F.R.S., father of W. H. Barlow, F.R.S., the latter of whom proposed the "theory of flexure," in an article in the *Civ. Eng. Jour.*, Vol. xxi., p. 113, assumes that $\phi = T$.

From the above it is inferred that the practical mean values of ϕ are:—

16,000 lbs. for cast-iron.
30,000 lbs. for wrought-iron.
8,000 lbs. for wood.

Example.—How much will a beam whose length is 12 feet, breadth 2 inches, depth 5 inches, sustain, if supported at its ends, and uniformly loaded over its whole length, and $C = 50{,}000$ lbs., $\phi = 30{,}000$ lbs., and coefficient of safety $\frac{1}{4}$?
Ans.—11,000 lbs. nearly.

142. BEAMS LOADED AT ANY NUMBER OF POINTS.—If the beam is loaded otherwise than has heretofore been supposed, it is only necessary to find the moment of all the forces in reference to the centre of a section and place the algebraic sum equal to the moments of resistance. Those which act in opposite directions will have contrary signs.

For instance, if a beam, AB, Fig. 70, rests upon two supports, and has weights P_1, P_2, P_3, etc., resting upon it at distances respectively of n_1, n_2, n_3, etc., from one support, and m_1, m_2, m_3, etc., from the other, the sum of the moments of the forces on any section C whose distance is x from the support A, is

$$V_1x - P_1(x - n_1) - P_2(x - n_2) \text{—etc.,}$$

to include all the terms of P in which n is less than x. This equals $\frac{1}{6} Rbd^2$ for rectangular beams.

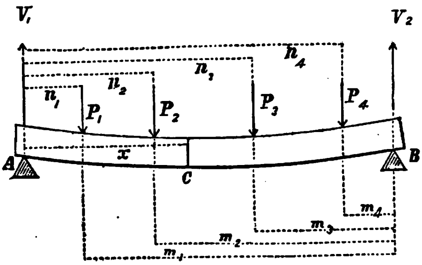

FIG. 70.

V_1, the reaction of one support, is readily found by taking the moments of all the external forces about B, and solving for V_1, thus:—

$$V_1 l = P_1 m_1 + P_2 m_2 + P_3 m_3 + \text{etc.}, = \Sigma Pm$$

$$\therefore V_1 = \frac{\Sigma Pm}{l}$$

$$\text{Similarly } V_2 = \frac{\Sigma Pn}{l}$$

$$\text{also, } V_1 + V_2 = P_1 + P_2 + P_3 + \text{etc.}, = \Sigma P.$$

143. A PARTIAL UNIFORM LOAD.—Let the beam be loaded uniformly over any portion of its length, as in Fig. 71.

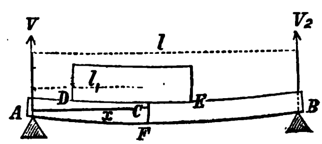

FIG. 71.

Let $l = AB$ = length of beam;
$2a = DE$ = length of the uniform load;
$x = AF$ = the distance to any section;
w = the load on a unit of length;
V = the reaction of the support A;
C the centre of the load;
$l_1 = AC$; $l_2 = CB$.

Then $AD = l_1 - a$, and $DF = x - l_1 + a$.

Load on $DF = w(x - l_1 + a)$,
" " $DE = 2wa$.

By the principle of moments

$$Vl = 2wa.\, l_2 \therefore V = 2wa\,\frac{l_2}{l}.$$

The moment of stress at F is

$$Vx - \tfrac{1}{2}w\,(x - l_1 + a)^2$$

$$\text{or } \frac{2wal_2}{l}\,x - \tfrac{1}{2}w\,(x - l_1 + a)^2 \quad \ldots\ldots(167)$$

That value of x which will make Equation (167) a maximum, gives the position of the dangerous section. Differentiate, place equal zero, and make $l_1 + l_2 = l$, and solve for x, and find

$$x = a\left(1 - \frac{2l_1}{l}\right) + l_1 \quad \ldots\ldots(168)$$

If $l_1 = \tfrac{1}{2}l,\ x = l_1$;
$l_1 < \tfrac{1}{2}l,\ x > l_1$;
$l_1 > \tfrac{1}{2}l,\ x < l_1$;

so that the maximum strain is at the centre of the loading only when the centre of the loading is over the centre of the beam; and in all other cases *it is nearer the centre of the beam than the centre of the loading is.*

The maximum strain is found by substituting the value of x Equation (168) in Equation (167).

The following interesting facts are also proved.

Let $AD = y \therefore a = l_1 - y$ which in Equation (168) reduces it to

$$x - l_1 = (l_1 - y)\left(1 - \frac{2l_1}{l}\right) \quad \ldots\ldots(168a)$$

which is a maximum for $y = 0$; hence so far as AD is concerned, Equation (168a) is a maximum when one end of the load is over the support, and for this case the equation becomes

$$x - l_1 = l_1\left(1 - \frac{2l_1}{l}\right)$$

which is a maximum for $l_1 = \tfrac{1}{4}\,l$ or $2l_1 = \tfrac{1}{2}l$, or the load must extend to the middle of the beam. Making $a = l_1 = \tfrac{1}{4}\,l$, and Equation (168) becomes

$$x = \tfrac{3}{8}l,$$

and these values of l_1 and x in Equation (167) give for the maximum moment of stress,

$$\tfrac{9}{128} w l^2 = \tfrac{9}{64} W l \ldots\ldots\ldots\ldots\ldots\ldots (169)$$

in which W is the load on half the beam.

Equation (167) gives the stress at the middle of the load, by making $a = l_1 = \frac{1}{4} l$ and $x = \frac{1}{4} l$. This gives $\frac{1}{8} Wl$ for the stress at the middle of the loading; hence, the maximum stress is $1\frac{1}{8}$ times the stress at the middle of the loading when the load extends from the one support to the middle of the beam.

144. GENERAL FORMULA.—The preceding methods are easily understood, and are perhaps sufficient for the more simple cases; but for the purposes of analysis a general formula is better, by means of which a direct analytical solution may be made for special cases.

Let R = *the modulus of rupture*, as explained in Article 120;
x and u horizontal coördinate axes, the former coinciding with the axis of the beam, and y a vertical axis;
Then $R\,du\,dy$ = the resistance of a fibre which is most remote from the neutral axis;
Let d_1 = distance between the neutral axis and the most remote fibre; then, according to the common theory, since the strains vary as the distance from the neutral axis

$$d_1 : y :: Rdudy : \text{resistance of any fibre} = \frac{R}{d_1} ydydu$$

$$\therefore \frac{R}{d_1} y^2\, dydu = \text{the } \textit{moment} \text{ of resistance of any fibre,}$$

and the sum of all the moments of resistance of any section is

$$\frac{R}{d_1} \int\int y^2 dy\, du = \frac{R}{d_1} I$$

which is called the *moment of rupture*, and must equal the sum of the moments of the straining forces;

$$\therefore \Sigma Px = \frac{R}{d_1} I \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (170)$$

The second member of this equation involves the character of the material (R) and the form of the transverse sections $(\frac{I}{d_1})$; the latter of which may be determined by analysis, and the former by experiment. The second member shows that for economy the material should be removed as much as possible from the neutral axis.

Let R' = the *strain* on a unit of fibres at a distance d_1 from the neutral axis, then

$$\Sigma Px = \frac{R'}{d_1} I \qquad (170a).$$

By comparing Equations (170a) and (49) we see that

$$\frac{E}{\rho} = \frac{R'}{d_1} \qquad (171)$$

which is true so long as the strain R' does not exceed the elastic limit.

145. LET THE BEAM BE RECTANGULAR, b the breadth, and d the depth, as in Fig. 61,

$$\text{Then } I = 4 \int_0^{\frac{1}{2}b} \int_0^{+\frac{1}{2}d} y^2 dy\, du = \tfrac{1}{12} bd^3$$

$$d_1 = \tfrac{1}{2} d$$

$$\therefore \frac{R}{d_1} I = \tfrac{1}{6} R\, bd^2 \text{ which is the same as expression (145).}$$

146. IF THE SIDES OF THE BEAM ARE INCLINED to the direction of the force, as in Fig. 72, let i be the inclination of the side to the horizontal; then

$$I = \tfrac{1}{12} bd\,(d^2 \sin^2 i + b^2 \cos^2 i)^*$$

$$d_1 = \tfrac{1}{2} d \sin i + \tfrac{1}{2} b \cos i$$

$$\therefore R \frac{I}{d_1} = \tfrac{1}{6} Rbd \left[\frac{d^2 \sin^2 i + b^2 \cos^2 i}{d \sin i + b \cos i}\right] \qquad (172).$$

* See Appendix II.

This expression has an algebraic minimum,* but not an *algebraic* maximum. By inspection, however, we find that the practical maximum is found by making $i = 90°$, if d exceeds b. Hence, a rectangular beam is strongest when its broad side is parallel to the direction of the applied forces.

Hence, the braces between joists in flooring, as in Fig. 73, not only serve to transmit the stresses from one to another, but also to strengthen them by keeping the sides vertical.

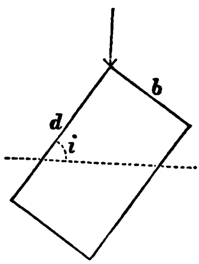

FIG. 72.

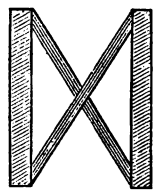

FIG. 73.

If $i = 90°$, Equation (172) becomes $\frac{1}{6}Rbd^2$.......(173).
If $b = d$ and $i = 45°$, Equation (172) reduces to

$$\frac{Rd^3}{6\sqrt{2}} \qquad \text{.................(174)}$$

(which is the same as Expression (163)),
and if $b = d$, and $i = 0°$ or $90°$, it becomes

$$\tfrac{1}{6}Rd^3.$$

Hence, the strength of a square beam having a side vertical is to the strength of the same beam having its diagonal vertical, as

$$1 : \sqrt{\tfrac{1}{2}},$$

or $\sqrt{2}$ to 1 or as 7 to 5 nearly.

In establishing Equation (172) it was assumed that the *neutral surface* was perpendicular to the direction of the applied forces, which is not strictly true unless the forces coincide with the diagonal; for in other cases there is a stronger tendency to

* See an article by the author in the *Journal of Franklin Institute*, Vol. LXXV., p. 260.

deflect sidewise than in the direction of the depth. In this case, as soon as the beam is bent there is a tendency to torsion. Both these conditions make the beam weaker than when the sides are vertical. If the tendency to torsion be neglected, the case may be easily solved; but the result shows the advantage of keeping the sides vertical.

147. THE STRONGEST RECTANGULAR BEAM which can be cut from a cylindrical one has the breadth to the depth as 1 to $\sqrt{2}$, or nearly as 5 to 7.

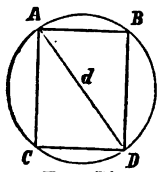

FIG. 74.

Let $x = AB =$ the breadth,
$y = AC =$ the depth, and
$D = AD =$ the diameter.

Then,

$$y^2 = D^2 - x^2$$

and Expression (173) becomes

$$\tfrac{1}{6}Kxy^2 = \tfrac{1}{6}Rx\,(D^2 - x^2),$$

which by the Differential Calculus is found to be a maximum for

$$x = D\sqrt{\tfrac{1}{3}} \therefore y = D\sqrt{\tfrac{2}{3}}$$

$$\therefore x : y :: 1 : \sqrt{2} \text{ or nearly as 5 to 7.}$$

Examples.—How much stronger is a cylindrical beam than the strongest rectangular one which can be cut from it?

(For the strength of a cylindrical beam, see Equation (180).)

Ans.—About 53 per cent.

How much stronger is the strongest rectangular beam that can be cut from a cylindrical one, than the greatest square beam which can be cut from it?

148. TRIANGULAR BEAMS.—If the base is perpendicular to the neutral axis, as in Fig. 75;

Let $d = AD =$ the altitude, and
$b = BC =$ the base.

Take the origin of coördinates at the centre of gravity of the triangle, y vertical and u horizontal.

Then, by similar triangles,

$$\tfrac{1}{2}b : y :: d : \tfrac{2}{3}d + u$$

$$\therefore y = \frac{\tfrac{1}{3}db + \tfrac{1}{2}bu}{d} \quad \therefore du = \frac{2d}{b}dy$$

$$\therefore I = 2\int\int y^2 dydu = 2\int_0^{\frac{1}{2}b} \tfrac{1}{3}y^3 du = \tfrac{1}{48}db^3.$$

We also have

$$d_1 = \tfrac{1}{2}b;$$

$$\therefore R\frac{I}{d_1} = \tfrac{1}{24}Rdb^2 = \tfrac{1}{12}RAb \qquad (175)$$

in which A is the area of the triangle.

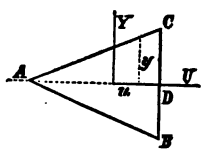

FIG. 75.

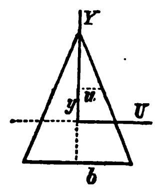

FIG. 76.

If the base is parallel to the neutral axis, as in Fig. 76, then, by similar triangles,

$$d : \tfrac{1}{2}b :: \tfrac{2}{3}d - y : u$$

$$\therefore u = (\tfrac{2}{3}d - y)\frac{b}{2d}$$

$$\therefore I = 2\int\int y^2 dydu = 2\int_{-\frac{1}{3}d}^{+\frac{2}{3}d} y^2 u dy$$

$$= \frac{b}{d}\int_{-\frac{1}{3}d}^{+\frac{2}{3}d} (\tfrac{2}{3}d - y)\, y^2 dy = \tfrac{1}{36}bd^3 \ (*)$$

We also have

$$d_1 = \tfrac{2}{3}d$$

---

* This is more easily solved by taking the moment about an axis through the vertex and parallel to the base, and using the formula of reduction. See Appendix.

$$\therefore R\frac{I}{d_1} = \tfrac{1}{24}Rbd^2 = \tfrac{1}{12}RAd \ldots\ldots\ldots\ldots\ldots\ldots (176).$$

Expressions (173) and (175) show that a triangular beam which has the same area and depth as a rectangular one, is only half as strong as the rectangular one.

Some authors have said that a triangular beam is twice as strong with its apex up as with it down, but this is not always the case. If the ultimate resistance of the material is the same for tension as for compression, the beam will be equally strong with the apex up or down.

If the beam is made of cast-iron, and supported at its ends, it will be about 6 times as strong with the apex up as down; but if the beam be fixed at one end, and loaded at the free end, it will be about 6 times as strong with the apex down as with it up.

149. **TRAPEZOIDAL BEAM.**—*Required the strongest trapezoidal beam which can be cut from a given triangular one.**

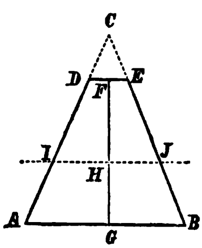

FIG. 77.

Let ABC be the given triangle,
$ABED$ the required trapezoid,
$d = CG =$ the longest altitude,
$b = AB$, $d_1 = FH$, $w = CF$,
$z = CH = d_1 + w$, and $v = DE$.

IJ is the neutral axis of the trapezoid, which passes through its centre of gravity H. We may then find:—

$$d_1 = \tfrac{1}{3}\frac{d}{b} \times \frac{2b^2 - bv - v^2}{b + v}$$

$$I = \tfrac{1}{36}\frac{d^3}{b^3}\left[\frac{b^5 + b^4v - 8b^3v^2 + 8b^2v^3 - bv^4 - v^5}{b + v}\right]$$

$$\therefore R\frac{I}{d_1} = \tfrac{1}{12}R\frac{d^2}{b^2}\left[\frac{b^5 + b^4v - 8b^3v^2 + 8b^2v^3 - bv^4 - v^5}{2b^2 - bv - v^2}\right] \quad (177)$$

which is to be a maximum. By the Calculus we find, after reduction, that

* See an article by the author in the *Journal of Franklin Institute*, Vol. XLI., third series, p. 198.

$$v^3 + 5bv^2 + 7b^2v - b^3 = 0,$$

for a maximum, which solved gives

$$v = 0.13093b \text{ or } 0.13b \text{ nearly, and hence}$$

$$w = 0.13093d \text{ or } 0.13d \ldots\ldots\ldots\ldots\ldots (178)$$

which substituted in (177) gives

$$R\frac{I}{d_1} = 0.545625\,\frac{Rd^2b}{12} \ldots\ldots\ldots\ldots (179).$$

Dividing Equation (179) by Equation (176) gives 1.09125; hence from (178) and (179) we infer that *if the angle of the prism be taken off* 0.13 *of its depth, the remaining trapezoidal beam will be* 1.091 *times as strong as the triangular one, which is a gain of over* 9 *per cent.*

In order to explain this paradox it must be granted that the condition does not require that the beam shall be broken in two, but that a fibre shall not be broken—in other words, the beam shall not be fractured. The greatest strain is at the edge, where there is but a single fibre to resist it; but, after a small portion of the edge is removed, there are many fibres along the line *DE*, each of which will sustain an equal part of the greatest strain.

If the triangular beam were loaded so as to just commence fracturing at the edge, the load might be increased 9 per cent. and increase the fracture to only thirteen-hundredths of the depth; but if the load be increased 10 per cent. it will break the beam in two.

These results are independent of the material of which the beam is made. If the beam be cut off $\frac{1}{8}$ the depth, its strength is found from Equation (177) to be

$$0.465608\,\frac{Rbd^2}{12},$$

which is 0.93101 of Equation (176).

Mr. Couch found* for the mean of seven experiments on triangular oak beams of equal length, that they broke with 306 pounds. The mean of two experiments on trapezoidal oak

* See Barlow's *Strength of Materials.*

beams, made from triangular beams of the same size as in the preceding experiments, by cutting off the edge one-third the depth when the narrow base was upward, was 284.5 pounds. This differs by less than half a pound of 0.931 times 306 pounds.

150. CYLINDRICAL BEAMS.—The moment of inertia of a circular section in which r is the radius, is

$$I = 2\iint y^2 dy du = \tfrac{2}{3}\int y^3 du = \tfrac{2}{3}\int_{-r}^{+r} (r^2 - u^2)^{\frac{3}{2}} du = \tfrac{1}{4}\pi r^4$$

$$d_1 = r;$$

$$\therefore \frac{RI}{d_1} = \tfrac{1}{4}R\pi r^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (180)$$

If polar coördinates are used, we have

$$du dy = \rho d\rho d\phi,$$

where ρ is a variable radius and ϕ a variable angle.

Also $y = \rho \sin \phi$

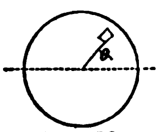
FIG. 78.

$$\therefore I = \iint y^2 dy du = \int_0^r \int_0^{2\pi} \rho^3 \sin^2 \phi d\rho d\phi$$

$$= \tfrac{1}{4}r^4 \int_0^{2\pi} \tfrac{1}{2}(1 - \cos 2\phi)\, d\phi = \tfrac{1}{4}\pi r^4, \text{ as before.}$$

For a circular annulus we have

$$R\frac{I}{d_1} = R\frac{\pi}{4r}(r^4 - r_1^4).$$

By comparing Equations (180) and (145) we see that the strength of a cylindrical beam is to that of a circumscribed rectangular one as $\frac{\pi}{32} : \frac{1}{6}$, or as 0.589 + : 1.

Also the strength of a cylindrical beam is to that of a square one of the same area as $\frac{1}{8}RAd'$ to $\frac{1}{6}RAd$ (d' being the diameter of the circle),

$$\text{or as } 1 : \left(\tfrac{4}{3}\frac{d}{d'} = \tfrac{2}{3}\sqrt{\pi}\right) \text{ or as } 1 : 1.18 \text{ nearly.}$$

It may be shown in the same sense as explained in the preceding article, that if a thin segment be removed from the upper and lower sides of the beam it will be stronger.

151. ELLIPTICAL BEAMS.

Let b = the conjugate axis, and
d = the transverse axis; then
if d is vertical (Fig. 79), we have
$I = \frac{1}{64}\pi b d^3$ and $d_1 = \frac{1}{2}d$.

If b is vertical (Fig. 80), we have

$I = \frac{1}{64}\pi b^3 d$ and $d_1 = \frac{1}{2}b$.

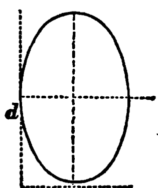

FIG. 79.

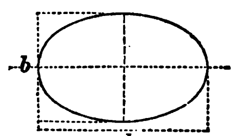

FIG. 80.

152. PARABOLIC BEAMS.

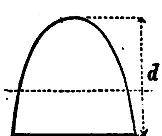

FIG. 81.

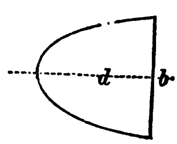

FIG. 82.

If b = the base, and
d = the height of the parabola, and
if d is vertical (Fig. 81), we have
$I = \frac{8}{175}bd^3$, and $d_1 = \frac{3}{5}d$.

If b is vertical (Fig. 82), then
$I = \frac{1}{30}b^3 d$, and $d_1 = \frac{1}{2}b$.

153. According to Barlow's Theory we have

$$\frac{T'}{d_1}\left[\int\int y^2 dydu\right] + \phi\int\int ydydu = \Sigma Px \qquad (181)$$

which must be integrated between the proper limits to include the whole section.

If the neutral axis is at the centre of the sections, and the beam is rectangular, we have

$$\frac{T}{d_1 = \frac{1}{2}d}\left[\int_0^b\int_{-\frac{1}{2}d}^{+\frac{1}{2}d} y^2 dydu\right] + 2\phi\int_0^{+\frac{1}{2}d}\int_0^b ydydu = \Sigma Px,$$

which reduced gives

$$\tfrac{1}{6}Tbd^2 + \tfrac{1}{4}\phi bd^2 = \tfrac{1}{12}[2T+3\phi]bd^2;$$

hence, if ϕ has any ratio to T, the *law of resistance* in solid rectangular beams is the same as for the common theory only.

If $\phi = T$, this becomes

$$\tfrac{5}{12}Tbd^2.$$

154. Oblique Strains.—If the force be inclined to the axis, as in Fig. 83, let θ = the angle which P makes with the axis of the beam.

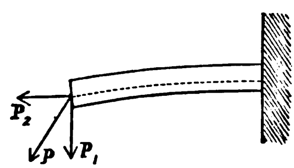

Fig. 83.

Then $P_2 = P\cos\theta$ = longitudinal component,
$P_1 = P\sin\theta$ = normal component.

If K = the transverse section, then

$\dfrac{P\cos\theta}{K}$ = the tension or compression upon a unit of section due to the direct pull or push. This tends directly to diminish the tabular value of R in the formula. If the beam be fixed at one end and P be applied at the free end, as in Fig. 83, the equation of moments becomes

$$Px\sin\theta = \left(R - \frac{P\cos\theta}{K}\right)\frac{I}{d_1}$$

which for rectangular beams becomes

$$P x \sin\theta = \left(R - \frac{P\cos\theta}{K}\right)\frac{bd^2}{6} \dots\dots\dots\dots (182)$$

in which θ is always acute.

This solution does not recognize any deflection. If the direction of P_2 does not intersect the neutral axis at the fixed end it will have a moment.

If flexure is considered, we find the strain upon the most remote fibre from the neutral axis at the fixed end, to which add the strain due to a direct pull (or push), which sum should not exceed the tabular value of R.

From Equation (171) we have

$$R' = \frac{E}{\rho} d_1$$

which is the strain on a unit of the extreme fibres.

From Equation (130) we have

$$\frac{E}{\rho} = E\frac{d^2y}{dx^2} = E(-p^2x + q^2y)$$

$$= E(-p^2l + q^2\Delta)$$

at the fixed end where the strain is evidently a maximum. From the Equations following (130) we find

$$q^2\Delta = \frac{p^2(e^{ql} - e^{-ql})}{q(e^{ql} + e^{-ql})} + p^2l$$

which substituted above gives

$$R' = \frac{d_1 E p^2 (e^{ql} - e^{-ql})}{q\ (e^{ql} + e^{-ql})}$$

$$= \frac{d_1 P \sin\theta\ (e^{ql} - e^{-ql})}{Iq\ \ (e^{ql} + e^{-ql})}$$

$$\therefore R = \frac{P_2}{K} + \frac{d_1 P_1}{Iq}\left(\frac{e^{ql} - e^{-ql}}{e^{ql} + e^{-ql}}\right)$$

$$\therefore P_1 = \frac{qIR}{d_1}\left(1 - \frac{P_2}{KR}\right)\left(\frac{e^{ql} + e^{-ql}}{e^{ql} - e^{-ql}}\right)$$

In the solution thus far we have supposed that rupture takes place on the side of tension, but if it should take place on the compressed side, we would have

$$P_1 = \frac{qIR}{d_1}\left(1 + \frac{P_2}{KR}\right)\left(\frac{e^{ql} + e^{-ql}}{e^{ql} - e^{-ql}}\right)$$

The total load, P, which the beam can sustain in these cases can be found only by a series of approximations, since P_2 and q both involve P.

The solution of the case shown in Fig. 83a, when flexure is not

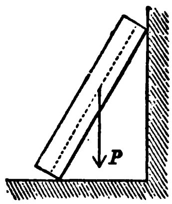

FIG. 83a.

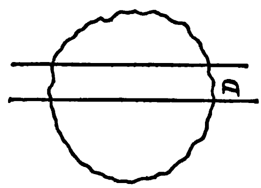

FIG. 83b.

considered, is given in "Bridges and Roofs," p. 20. If flexure is considered, the reaction at the ends will be treated as the oblique forces, and the solution made substantially as in the preceding case.

155. POSITION OF THE NEUTRAL AXIS FOR MINIMUM STRENGTH.

Let I_0 = the moment of inertia of the section when the axis passes through the centre of gravity of the section,

I = the moment of inertia of the same section about an axis parallel to the former,

D = the distance between the axes,

A = the area of the section, and

a_1 = the ordinate to the most remote fibre from the centre,

Then $D + a_1$ = the ordinate to the most remote fibre from the second axis, and

$$R\frac{I}{d_1} = \frac{I_0 + AD^2}{a_1 + D} \qquad (181c)$$

which is a minimum for

$$D = a_1\left[-1 \pm \sqrt{1 + \frac{I_0}{Aa_1^2}}\right]$$

One of the roots is positive and less than d_1, and the other is negative and greater than d_1, but both give an algebraic minimum.

For a rectangle $A = bd$, $I_0 = \frac{1}{12}bd^3$, and $a_1 = \frac{1}{2}d$.

$$\therefore D = 0.07732d \text{ or } -1.07732d.$$

Using the positive value, we have

$$R\frac{I}{d_1} = 0.1547Rbd^2;$$

which is only 0.9282 of the strength when the axis passes through the centre.

If the sections are circular

$$D = 0.11807r$$

and

$$\frac{RI}{d_1} = 0.7415Rr^3,$$

which is 0.9441 of the strength when the axis passes through the centre.

Has this analysis any physical signification? Being entirely independent of the character of the material, it does not explain the difference between the values of R and T or C. So far as the analysis is concerned there is nothing to determine which way the neutral axis will move from the centre.

In some cases, *practically*, we might have $d_1 = a_1 - D$; in which case we have for a minimum

$$D = a_1\left[1 \pm \sqrt{1 + \frac{I_0}{Aa_1^2}}\right]$$

which for rectangular beams gives

$$D = 1.077d \text{ or } -0.077d.$$

CHAPTER VII.

BEAMS OF UNIFORM RESISTANCE.

156. GENERAL EXPRESSION.—If beams are so formed that they are equally liable to break at every transverse section, they are *beams of uniform resistance*, and are generally called *beams of uniform strength.* The former term is preferable, because it applies with equal force to all strains less than that which will produce rupture. In such a beam the strain on the fibre most remote from the neutral axis is uniform throughout the whole length of the beam. The analytical condition, according to the common theory, is: *The sum of the moments of the resisting forces must vary directly as the sum of the moments of the applied forces;* hence Equation (171) is applicable; or

$$\Sigma Px = \frac{RI}{d_1} \dots\dots\dots\dots (182)$$

which must be true for all values of x. In addition to this the transverse shearing strain must be provided for. To obtain practical results it is necessary to consider

PARTICULAR CASES.

157. BEAMS FIXED AT ONE END AND LOADED AT THE FREE END.—*Required the form of a beam of uniform resistance when it is fixed at one end and loaded at the free end.*

1st. Let the sections be rectangular, and

$y =$ the variable depth, and
$u =$ the variable width.

Then $I = \frac{1}{12}uy^3$ (see Equation (51)),
$d_1 = \frac{1}{2}y$, and
$\Sigma Px = Px =$ the variable load.*

* For ΣPx use the general moments as given in the table in Article 109, so far as they are applicable.

Hence Equation (182) becomes

$$Px = \tfrac{1}{6}Ruy^2 \dots\dots\dots\dots (183)$$

a. Let the breadth be constant; or $u = b$; then (183) becomes

$$Px = \tfrac{1}{6}Rby^2 \dots\dots\dots\dots (184)$$

which is the equation of a parabola, whose axis is horizontal and parameter is $\frac{24P}{Rb}$. See Fig. 84.

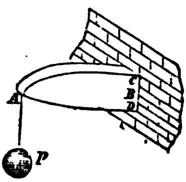

FIG. 84.

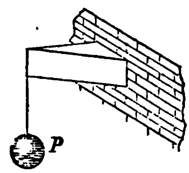

FIG. 85.

b. Suppose that the depth is constant, or $y = d$. Then (183) becomes

$$Px = \tfrac{1}{6}Rd^2u \dots\dots\dots\dots (185)$$

which is the equation of a straight line; hence the beam is a wedge, as in Fig. 85.

c. If the sections are rectangular and similar, then

$$u : y :: b : d$$

$$\therefore u = \frac{b}{d}y;$$

and Equation (183) becomes

$$Px = \frac{Rb}{6d}y^3,$$

which is the equation of a cubical parabola.

2d. Let the sections be circular. Then $I = \frac{1}{64}\pi y^4$ Equation (52), in which y is the diameter of the circle), and $d_1 = \frac{1}{2}y$; hence (182) becomes

$$Px = \tfrac{1}{32}R\pi y^3,$$

which is also the equation of a cubical parabola, as shown in Fig. 86.

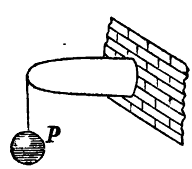

FIG. 86.

3d. Let the transverse sections be rectangular, and I constant, the breadth and depth both being variable, then Equation (182) becomes

$$Px = R\frac{(\frac{1}{12}uy^3)}{\frac{1}{2}y} = R\frac{c}{6y}\text{.............}(186)$$

in which c is a constant, $= bd^3$, b and d being the breadth and depth at the fixed end. Equation (186) is the equation of the vertical longitudinal sections, and is the equation of an hyperbola referred to its asymptotes. See Fig. 87. If the value of y from this equation be substituted in the Equation $uy^3 = c$, it gives

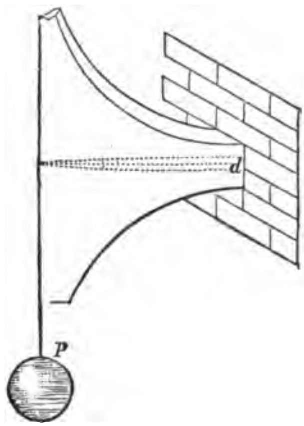

FIG. 87.

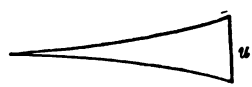

FIG. 88.

$$u = \frac{216P^3x^3}{R^3b^2d^6}\text{........................}(187)$$

which is the equation of the horizontal longitudinal sections; hence they are cubical parabolas, as in Fig. 88. For x and $u = 0$, $y = \infty$, and for $x = l$, $u = b = \frac{216P^3l^3}{R^3b^2d^6} \therefore b = \frac{6Pl}{Rd^2}$

4th. If the breadth is the nth power of the depth, and the sections are rectangular, then $u = y^n$, and Equation (183) becomes

$$Px = \frac{1}{6}Ruy^2 = \frac{1}{6}Ry^{n+2},$$

which is the general equation of parabolas.

158. BEAMS FIXED AT ONE END AND UNIFORMLY LOADED.—*Required the form of a beam of uniform resistance* *when it is fixed at one end and uniformly loaded over its whole length; the weight of the beam being neglected.*

The origin of coördinates being still at the free end, we have

$wx =$ the load on a length x, and

$\frac{1}{2}wx^2 =$ the moment of the load (Equation (53)).

Hence, for rectangular sections, Equation (182) becomes

$$\tfrac{1}{2}wx^2 = \tfrac{1}{6}Ruy^2 \qquad (188)$$

a. If the breadth is constant, or $u = b$ in (188), it becomes

$$\tfrac{1}{2}wx^2 = \tfrac{1}{6}Rby^2,$$

which is the equation of a straight line, and hence the beam will be a wedge, as in Fig. 89.

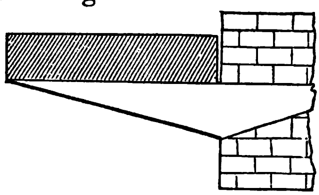

FIG. 89.

b. Let the depth be constant; or $y = d$ in (188)

$$\therefore \tfrac{1}{2}wx^2 = \tfrac{1}{6}Rd^2u;-$$

a parabola whose axis is perpendicular to the axis of the beam, as in Fig. 90.

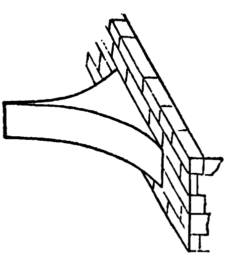

FIG. 90.

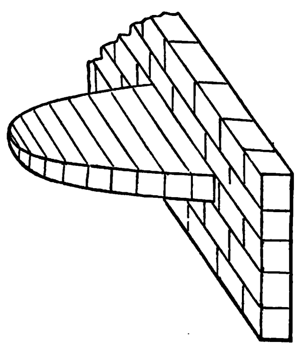

FIG. 91.

c. Let the sections be similar;—

then $d : b :: y : u = \frac{b}{d}y,$

$\therefore$ Equation (188) becomes $\frac{1}{2}wx^2 = \frac{1}{6}R\frac{b}{d}y^3$;—

a semi-cubical parabola, as in Fig. 91.

d. Let I be constant, or $\frac{1}{12}uy^3 = \frac{1}{12}bd^3$. Then Equation (182) becomes

$$\frac{1}{2}wx^2 = \frac{1}{6}R\frac{bd^3}{y};\text{—an hyperbola of the second order.}$$

159. PREVIOUS CASES COMBINED.—*Required the form of the beam of uniform resistance when it is fixed at one end and loaded uniformly, and also loaded at the free end.*

The moment of applied forces is $Px + \frac{1}{2}wx^2$; hence Equation (182) becomes, for rectangular beams,

$$Px + \frac{1}{2}wx^2 = \frac{1}{6}Ruy^2.$$

Hence, if the depth is constant, $Px + \frac{1}{2}wx^2 = \frac{1}{6}Rud^2$;—a parabola.

Hence, if the breadth is constant, $Px + \frac{1}{2}wx^2 = \frac{1}{6}Rby^2$;—a hyperbola.

Hence, if the sections are similar, $Px + \frac{1}{2}wx^2 = \frac{1}{6}R\frac{b}{d}y^3$;—a semi-cubical parabola.

160. WEIGHT OF THE BEAM CONSIDERED.—*Required the form of the beam of uniform resistance when the weight of the beam is the only load; the beam being fixed at one end and free at the other.*

a. Let the sections be rectangular and the breadth constant.

Let $x = AB$; Fig. 92,

$y = DC$,

$b =$ the breadth, and

$\delta =$ the weight of a unit of volume.

Then $\int ydx =$ the area ADC, and

$\delta b\int ydx =$ the weight of ADC;

the limits of integration being 0 and x.

If F is the centre of gravity of ADC; we have,

$$AF = \frac{\int xydx}{\int ydx}.$$

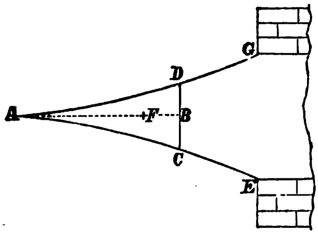

FIG. 92.

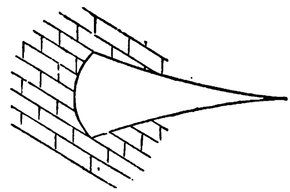

FIG. 93.

The moment of the applied forces is the weight of ADC multiplied by the distance $BF = x - AF$. Hence, Equation (182) becomes

$$\delta b \int y dx \left[x - \frac{\int xy dx}{\int y dx} \right] = \frac{1}{6} R b y^2,$$

which reduced gives

$$x^2 = \frac{2R}{\delta} y \ldots\ldots\ldots\ldots\ldots\ldots (189)$$

which is the equation of the common parabola, the axis being vertical.

b. Let the depth be constant. In a similar way we find

$$\delta d \int u dx \left[x - \frac{\int ux dx}{\int u dx} \right] = \frac{1}{6} R d^2 u.$$

This solved gives

$$x = -\sqrt{\frac{Rd}{6\delta}} \text{ Nap. log.} \left[\sqrt{\frac{Rd}{6\delta} C + u^2} - u \right] + C',$$

in which C and C' are constants of integration, and involve the position of the origin of coördinates and direction of the curve at a known point.

c. Let the beam be a conoid of revolution, as in Fig. 93. We have, as before (y being the radius of the circle),

$$\delta \int \pi y^2 dx \left[x - \frac{\int \pi y^2 x dx}{\int \pi y^2 dx} \right] = \frac{1}{4} \pi R y^3,$$

which reduced gives

$$x^2 = \tfrac{15}{2} \frac{R}{\delta} y \text{..............................(191)}$$

which is the equation of the common parabola.

d. Suppose, in the preceding cases, that an additional load, P, is applied at the free end.

Some of the equations which result from this condition cannot be integrated in finite terms, and hence the curves cannot be classified.

161. BEAMS SUPPORTED AT THEIR ENDS.

A. *Let the beam be supported at its ends and loaded at the middle point.*

For this case, Equation (182) becomes, for rectangular sections,

$$\tfrac{1}{2}Px = \tfrac{1}{6}Ruy^2 \text{........................(192)}$$

a. If the breadth is constant, we have

$$\tfrac{1}{2}Px = \tfrac{1}{6}Rby^2,$$

which is the equation of the common parabola.

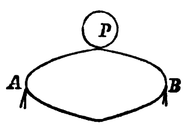

FIG. 94.

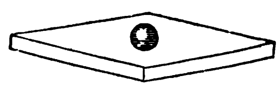

FIG. 95.

The beam consists of two parabolas, having their vertices, one at each support, as in Fig. 94.

b. If the depth is constant, we have

$$\tfrac{1}{2}Px = \tfrac{1}{6}Rd^2u \text{........................(193)}$$

a wedge, as in Fig. 95.

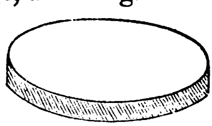

FIG. 96.

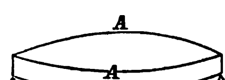

FIG. 97.

B. *If the beam is uniformly loaded*, we have from Equations (74) and (182),

$\frac{1}{2}w\ (lx - x^2) = \frac{1}{6}Ruy^2$—if rectangular, and if the breadth is constant, $\frac{1}{2}w\ (lx - x^2) = \frac{1}{6}Rby^2$....................(194)
an ellipse, Fig. 96.

If the depth is constant, $\frac{1}{2}w\ (lx - x^2) = \frac{1}{6}Rd^2u$, a parabola, Fig. 97.

C. *Let the beam have an uniform load and also an uniformly increasing load from one end to the other, as in* Fig. 98.

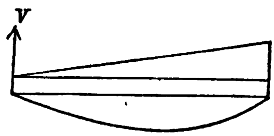

FIG. 98.

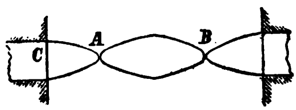

FIG. 99.

Let $W =$ the weight of the uniform load,
$W_1 =$ the weight of the uniformly increasing load, and
$V =$ the reaction of the support at the end which has the least load.

Then $V = \frac{1}{2}W + \frac{1}{3}W_1$.

Let x be reckoned from A, then the load on x is

$$\frac{W}{l}x + \frac{W_1}{l^2}x^2,$$

and the moment of this reaction and load on a section which is at a distance x from A is

$$(\tfrac{1}{2}W + \tfrac{1}{3}W_1)x - \frac{Wx^2}{2l} - \frac{W_1x^3}{3l^2} \ldots\ldots\ldots\ldots\ldots (195)$$

which equals $\frac{1}{6}R\,by^2$ for rectangular beams of uniform breadth. To find the point of greatest strain, make the first differential coefficient of (195), equal to zero. We thus find

$$\tfrac{1}{2}W + \tfrac{1}{3}W_1 - \frac{W}{l}x - \frac{W_1}{l^2}x^2 = 0.$$

If $W = 0$, this gives

$$x = \tfrac{1}{3}l\sqrt{3}.$$

When $W = 0$, this becomes the case of a fluid pressing against a vertical surface.

162. BEAMS FIXED AT THEIR ENDS.—If the beam is fixed at its ends and loaded at the middle with a weight, P, we have, from Equations (117) and (182), when the breadth is uniform,

$$\tfrac{1}{8}P(l - 4x) = \tfrac{1}{6}Rby^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (196)$$

which is the equation of a parabola. The beam really consists of four double parabolas with their vertices tangent to each other, as in Fig. 99. The vertices are $\frac{1}{4}l$ from the end.

If the load were uniform we would obtain, in a similar way, a beam composed of four wedges. These are direct deductions from the common theory.

This shows in a very marked degree the absurdity of not providing for the transverse shearing strain. All of the preceding cases show the same absurdity. The section being reduced to naught leaves no ability to resist the shearing strain. In a case like Fig. 99, it even prevents the equation of moments from being *practically* realized; for the resisting forces cannot be transmitted past the points A and B.

163. EFFECT OF TRANSVERSE SHEARING STRESS *on modifying the forms of the beams of uniform resistance.*

Take, for example, the case of a beam supported at its ends and uniformly loaded. The transverse shearing strain is

$$Ss = \tfrac{1}{2}wl - wx = \tfrac{1}{2}w(l - 2x),$$

which is the equation of a straight line, Fig. 100.

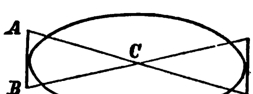

FIG. 100.

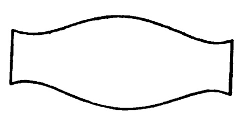

FIG. 101.

The double ordinate at the end is

$$\tfrac{1}{2}wl \div (b \times \text{modulus of shearing, } S,)$$

in which b is the breadth of the beam.

If the resistance to transverse shearing varies directly as the transverse section, then will the triangle ACB represent the vertical section of one-half of a beam of uniform strength when shearing alone is considered. This result is as absurd as the preceding.

Practically, the two cases may be combined by adding the ordinates of the line AC to those of the ellipse, the result being shown in Fig. 101.

Theoretically, I do not see how they can be combined, since the conditions upon which the equations are established are not only independent, but are not simultaneous.* Each condition furnishes a determinate equation. One is an equation of moments, and the other of forces. The *practical* solution above suggested, doubtless gives an excess of strength at all points, except at the ends and middle; for by increasing the depth we increase the moments of resistance, and probably add more than is necessary to resist the transverse shearing, since that is greatest near the neutral axis where the strain from moments is least.

164. UNSOLVED PROBLEMS.—Many practical problems in regard to the resistance of materials cannot be solved according to any known laws of resistance. Some of these have been solved experimentally, and empirical formulas have been deduced from the results of the experiments, which are sufficiently exact for practical purposes, within the range of the experiments. The resistance of tubes to collapsing, the strength of columns, and the proper thickness of the vertical web of rails, are such problems which have been solved experimentally. The following problems are of this class, and have not been solved. The first four are taken from the *Mathematical Monthly*, Vol. I., page 148.

1. Required a formula for the strength of a circular flat iron plate of uniform thickness, supported throughout its circumference and loaded uniformly.

---

* To illustrate, suppose it is required to find the radius of a sphere whose volume equals (numerically) the area of the surface; and whose diameter equals (numerically) the area of the hemisphere. The former gives $r = 3$, and the latter, $r = \frac{1}{\pi}$.

2. Required the strength of the same plate if the edges are bolted down.

3. Required the equation of the curve for each of the preceding cases, that they may have the greatest strength with a given amount of material.

4. In the preceding problems, suppose that the plate is square.

5. Required the form of a beam of uniform strength which is supported at its ends, the weight of the beam being the only load. Suppose, also, that it is loaded at the middle.

The latter part of this problem has received an approximate solution under certain conditions, as will be seen from the following experiments.

165. BEST FORM OF CAST-IRON BEAM AS FOUND EXPERIMENTALLY.—Cast-iron beams were first successfully used for building purposes by Messrs. Boulton & Watt. The form of the cross-section of the beams which they used is shown in Fig. 102. More recent experiments show that this is a good form, but not the best.

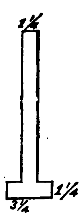

FIG. 102.

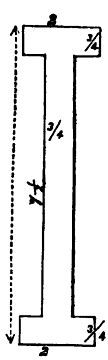

FIG. 103.

About 1822 Mr. Tredgold made an experiment upon a cast-iron beam of the form shown in Fig. 103, to determine its deflection. He recommended this form for beams.

Mr. Fairbairn has justly the credit of making the first series of experiments for determining the best *form of the beam.* These experiments were prosecuted by himself for a few years, beginning about 1822, and continued still later by Mr. Hodgkinson.

The experiments quickly indicated that the lower flange should be considerably the largest.

The following experiments were made by Mr. Hodgkinson (Fairbairn on *Cast and Wrought-Iron*, p. 11).

Fig. 104 shows the elevation and cross-section of a beam whose dimensions are as follows:—

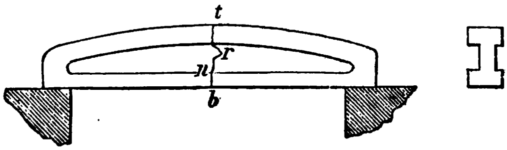

FIG. 104.

Area of top rib = 1.75 × 0.42 = 0.735 inches.
Area of bottom rib = 1.77 × 0.39 = 0.690 "
Thickness of vertical rib.......... 0.29 "
Depth of the beam...............5.125 "
Distance between the supports......54.00 "
Area of the whole section..........2.82 square inches.
Weight of the beam...............$36\frac{1}{4}$ pounds.
Breaking weight..................6,678 pounds.

The form of the fracture is shown at *b n r*. It broke by tension.

EXPERIMENT IV.

Dimensions.	*Inches.*
Thickness at A	= 0.32
" " B	= 0.44
" " C	= 0.47
" " FE	= 2.27
" " DE	= 0.52

Depth of the beam = 5.125

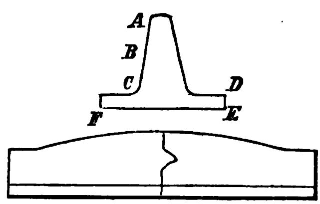

FIG. 105.

Area of the section = 3.2 square inches.
Distance between the supports = 54 inches.
Weight of casting = $40\frac{1}{2}$ pounds.
Deflection with 5,758 pounds = 0.25 inches.
" " 7,138 " = 0.37 "
Breaking weight 8,270 pounds.

Experiment 19.

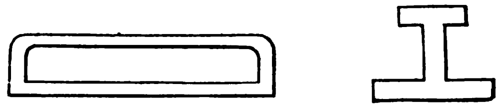

Fig. 106.

Dimensions in inches:—
Area of top rib = 2.33 × 0.31 = 0.72.
Area of bottom rib = 6.67 × 0.66 = 4.4.
Ratio of the area of the ribs = 6 to 1.
Thickness of vertical part = 0.266.
Area of section, 6.4.
Depth of beam, 5$\frac{1}{8}$.
Distance between the supports, 54 inches.
Weight of beam, 71 pounds.

This beam broke by compression at the middle of the length with 26,084 pounds.

It is probable that the neutral axis was very near the vertex *n*, or about $\frac{3}{4}$ the depth.

Experiment 21.

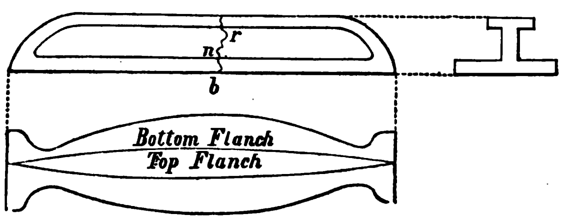

Fig. 107.

This was an elliptical beam, Fig. 107.
Dimensions in inches:—
Area of top rib = 1.54 × 0.32 = 0.493.
Area of bottom rib = 6.50 × 0.51 = 3.315.
Ratio of ribs, 6$\frac{1}{2}$ to 1.
Thickness of vertical part = 0.34.
Depth of beam, 5$\frac{1}{8}$.
Area of the section, 5.41.

Distance between supports, 54 inches.
Weight of beam, 70¾ pounds.
Broke at the middle by tension with 21.009 pounds.
Form of fracture $b\ n\ r$; $b\ n = 1.8$ inches.

As these beams have all the same depth and rested on the same supports, 4 feet 6 inches apart, their relative strengths will be *approximately* as the breaking weight divided by the area of the cross section.

In Experiment	1,	$6{,}678 \div 2.82 = 2{,}368$	lbs.	per square inch.
"	"	14,	$8{,}270 \div 3.2 = 2{,}584$	"	"
"	"	19,	$26{,}084 \div 6.4 = 4{,}075$	"	"
"	"	21,	$21{,}009 \div 5.41 = 3{,}883$	"	"

It is evident from these experiments, that when the vertical rib is thin, the area of the lower rib should be about 6 times that of the upper. In the 19th experiment it has already been observed that the beam broke at the top, and in the 21st it broke at the bottom, although the lower flange was larger in proportion to the upper than in the preceding case, and the comparison shows that they were about equally well proportioned. They should be so proportioned that they are equally liable to break at the top and bottom.

A beam proportioned so as to be similar to either of the two last forms above mentioned may be called a "TYPE FORM."

166. HODGKINSON'S FORMULAS *for the strength of cast-iron beams of the* TYPE FORM.

Let $W =$ the breaking weight in tons (gross).
$a =$ the area of bottom rib at the middle of the beam.
$d =$ the depth of the beam at the middle.
and $l =$ the distance between the supports.

Then according to Mr. Hodgkinson's experiments we have

$W = 26 \frac{ad}{}$ when the beam is cast with the bottom rib up, and

$W = 24 \frac{ad}{l}$ when the beam is cast on its side.

167. EXPERIMENTS ON T RAILS.—Experiments on T bars, supported at their ends and loaded at the middle, gave the following results:—*

Hot blast bar, rib upward, ⊥ broke with.......1,120 pounds.
" " " downward, T broke with.... 364 "
Cold blast bar, rib upward, ⊥ broke with......2,352 "
" " " downward, T broke with.... 980 "

The ratio of the strengths is nearly as 3 to 1, but according to the table in Article 47, we might reasonably expect a higher ratio. If a greater number of experiments would not have given a higher ratio, we would account for the discrepancy by supposing that the neutral axis moved before rupture took place, or that the ratio of the crushing strength to the tenacity is less for comparatively thin castings than for thick ones. It is known that the crushing strength of thin castings is proportionately stronger than thick ones. Hodgkinson found that for castings 2, 2½, and 3 inches thick, the *crushing strengths* were as 1 to 0.780 to 0.756; and Colonel James found a greater increase—being as 1 to 0.794 to 0.624. See also Article 41.

168. WROUGHT-IRON BEAMS.—The treacherous character of cast-iron beams has led to the introduction of solid wrought-iron ones. Special machinery and special processes of manufacture have been brought into use, by means of which they are quickly and cheaply made. They are usually of the double T (I) section.

169. A NOVEL AND PECULIARLY CONSTRUCTED FLOOR is here given as an illustration of the use of a plate (see Article 164, No. 4).† It was executed in Amsterdam, for a floor 60 feet square. The flooring consists of three thicknesses of 1½-inch boards. The first thickness is laid diagonally across the opening. The ends resting on the rebates of the wall-plates,

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* Mahan's *Civ. Eng.*, Wood's Ed., pp. 135 and 136; Barlow on the *Strength of Materials*, p. 183.

† Tredgold's *Principles of Carpentry*, 1870, p. 91.

and rising about 2½ inches higher in the middle of the room than at the sides. The second thickness is also laid diagonally, but square across the first, and the two well nailed together. The third thickness is laid parallel to the sides of the room, and the whole well nailed together. All the boards are grooved and tongued together, forming a floor 4½ inches thick. The strength of plates vary as the square of their thickness, and are equally strong to support a weight in the middle, whatever the extent of the bearing may be; but when the load is uniformly distributed, the strength varies inversely as the area of the space it covers.*

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* Emerson's *Mechanics*, sec. viii., prop. 73, cor. 5.

other end, and acting in the direction of a tangent to the arc of the path described by the free end.

As a unit of fibres cannot be placed so that all of them will be at a unit's distance from the axis, we must suppose that the resistance of a very thin annulus, which is at a unit's distance, is proportional to that of a unit of section. The area of an element is

$$\rho d\rho d\phi.$$

The resistance of an element which is at a unit's distance from the axis is G multiplied by its area; which expressed analytically is

$$G\rho d\rho d\phi,$$

and according to the first law

$G\rho^2 d\rho d\phi =$ the resistance of any fibre whose length is unity, to being twisted through an angle unity; and the moment of resistance $= G\rho^3 d\rho d\phi$ for an angle unity; and for any angle θ the moment is, according to the second law,

$$G\theta\rho^3 d\rho d\phi$$

and the total moment equals the moment of the applied force, or moments of the applied forces; hence

$$Pa = G\theta\iint\rho^3 d\rho d\phi = G\frac{a}{l}I_p,$$

where I_p is the polar moment of inertia of the section.

For circular sections $I_p = \int_0^r\int_0^{2\pi}\rho^3 d\rho d\phi = \frac{1}{2}\pi r^4$ (199)

$$\therefore G = \frac{2Pa}{\pi\theta r^4} = \frac{2Pal}{\pi\alpha r^4} \quad \ldots\ldots (200)$$

$$\text{or, } \theta = \frac{2Pa}{G\pi r^4}$$

172. THE VALUE OF THE COEFFICIENT G may be found from Equation (200). M. Cauchy found analytically on the condition that the elasticity of the material was the same in all directions, that $G = \frac{2}{5} E$.* M. Duleau found experimentally that G is less than $\frac{2}{5} E$, and nearly equal $\frac{1}{3} E$,† and M. Wertheim found $G = \frac{3}{8} E$ nearly.‡ M. Duleau's experiments gave the following mean values for G: ‡

	Value of G. Pounds
Soft iron	8,533,680
Iron bars	9,480,917
English steel	8,533,680
Forged steel (*very fine*)	14,222,800
Cast iron	2,845,600
Copper	6,209,670
Bronze	1,516,150
Oak	568,912
Pine	615,472

Example.—If an iron shaft whose length is 5 feet, and diameter 2 inches, is twisted through an angle of 7 degrees by a force $P = 5,000$ lbs., acting on a lever, $a = 6$ inches, required G. The 7 degrees is first reduced to arc by multiplying it by $\frac{\pi}{180}$, which gives $\alpha = \frac{7\pi}{180}$, and Eq. (200) gives,

$$G = \frac{2 \times 5000 \times 6 \times 60 \times 180}{(3.1416)^2 \times 7} = 9,697,000 \text{ lbs.}$$

173. TORSION PENDULUM.—If a prism is suspended from its upper end, and supports an arm at its lower end, and two weights each equal $\frac{1}{2} W$ are fixed on the arm at equal distances from the prism, and the prism be twisted and then left free to move, the torsional force will cause an angular movement of the arm until the fibres are brought to their normal position, after which they will be carried forward into a new position by the inertia of the moving mass in the weights $\frac{1}{2} W$ until the torsional resistance of the prism arrests their movements, after which they will reverse their movement, and an oscillation will result.

Equation (200) readily gives:

$$P = \frac{\pi G r^4}{2la^2} \ (a\alpha)$$

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* See Chapter IX.

† *Résistance des Matériaux*, Morin, p. 461.

‡ *L'Engineer*, 1858, p. 52.

from which it appears that the torsional force P varies as the space (aa) over which it moves.

It is a principle of mechanics that the moving force varies directly as the product of the moving mass multiplied by the acceleration. Hence, if $x = (aa)$, the variable space, $t =$ the variable time, $M =$ the mass moved. and observing that t and x are inverse functions of each other, and the above principle of mechanics gives the following equation (neglecting the mass of the prism):—

$$M\frac{d^2x}{dt^2} = -P = -\frac{\pi Gr^4}{2la^2}x.$$

Multiplying both members by the dx, gives

$$\frac{W}{g}\,\frac{dxd^2x}{dt^2} = -\frac{\pi Gr^4}{2la^2}xdx,$$

where W is the weight of the mass moved, and g is the acceleration due to gravity. The oscillations commence at the extremity of an arc whose length is s, at which point the velocity is zero. The integral of the last equation between the limits s and x is

$$\frac{dx^2}{dt^2} = \frac{\pi Ggr^4}{2Wla^2}(s^2 - x^2).$$

A second integral gives

$$t = \sqrt{\frac{2Wla^2}{\pi Ggr^4}}\left[\sin^{-1}\frac{x}{s}\right]_0^s = \sqrt{\frac{\pi Wla^2}{2Ggr^4}}$$

which is the time of half an oscillation. For a whole oscillation:

$$2t = T = \frac{a}{r^2}\sqrt{\frac{2\pi}{Gg}lW}.$$

This is essentially the theory of Coulomb's torsion pendulum. A torsion pendulum was used by Cavendish in 1778 to determine the density of the earth. (See *Royal Philosophical Transactions:* London, Vol. 18, p. 388.) He found the mean density of the earth by this method to be 5.48 times that of water, or according to Hutton's revision, 5.42.

Reich, by aid of a mirror apparatus, afterwards found it to be 5.43. Bailey found by experiments on a larger scale 5.675. Reich repeated his experiments and found 5.583. Other methods gave a value somewhat larger than these, but the mean result shows that the mean density of the earth is about 5½ times that of water.—See Bailey's *Experiments*, London, 1843.

174. RUPTURE BY TORSION.—The resistance which a bar offers to a twisting force is a *torsional shearing resistance*, and in regard to rupture, the equation of equilibrium is founded upon the following principles:—

1st. The strain upon any fibre varies directly as its distance from the axis of torsion; and

2d. The sum of the moments of resistance of the fibres equals the sum of the moments of the twisting forces.

Let J = the MODULUS OF TORSION, that is, the ultimate resistance to torsion of a unit of the transverse section which is most remote from the axis of torsion. It is the ultimate shearing resistance to torsion, but may be used for any shearing *strain* which is less than the ultimate,

d_1 = the distance of the most remote fibre from the axis of torsion,

$f(\rho, \phi)$ = the equation of the section,

P = the twisting force, and

a = the lever arm of P.

Ip = the polar moment of inertia of a section.

Then $\rho d\rho d\phi = dA$ = the area of an element of the section;

$J\rho d\rho d\phi$ = the shearing strain of the most remote element; and, by the first principle given above,

$\frac{J}{d_1}\rho d\rho d\phi$ = the shearing strain of any element, which is at a unit's distance from the axis of torsion, and from the same principles we have

$\frac{J}{d_1}\rho^2 d\rho d\phi$ = the shearing strain of any element, and this, multiplied by the distance of the element, ρ, from the axis, gives

$\frac{J}{d_1}\rho^3 d\rho d\phi$ = the moment of resistance to torsion.

Hence, according to the second principle we have

$$Pa = \frac{J}{d_1}\int\int \rho^3 d\rho d\phi = \frac{J}{d_1}\int \rho^2 dA. = \frac{J}{d_1} Ip \ldots\ldots (201)$$

For circular sections, we have already found, Eq. (199),

$$Ip = \tfrac{1}{2}\pi r^4.$$

For square sections, whose sides are b, we may find *

$$Ip = \tfrac{1}{6} b^4, \text{ and } d_1 = b\sqrt{\tfrac{1}{2}}.$$

* We have $\int \rho^2 dA = \int \left(x^2 + y^2\right) dA = \int x^2 dA + \int y^2 dA$, that is, the polar moment equals the sum of the rectangular moments, the origin being the same

175. PRACTICAL FORMULAS.—Equations (199) and (201) give for cylindrical pieces, observing that $d_1 = r$,

$$Pa = \tfrac{1}{2}\pi J r^3 \therefore J = \frac{2\,Pa}{\pi r^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots (202)$$

If cylindrical pieces are twisted off by forces which form a *couple*, and P, a, and r measured, the value of J may be found from Equation (202). Cauchy found $J = \frac{4}{5}R$,* which is considered sufficiently exact when a proper coefficient of safety is used. Calling $J = 25{,}000$ pounds for iron, and using about a five-fold security; and $J = 8{,}000$ pounds for wood, and using about a ten-fold security, and we may use for

$$\left.\begin{array}{ll}\text{ROUND IRON SHAFTS (wrought or cast), diameter} & = \frac{1}{10}\sqrt[3]{Pa} \\ \text{SQUARE IRON SHAFTS (wrought or cast), side of the square} & = \frac{1}{11}\sqrt[3]{Pa} \\ \text{SQUARE WOODEN SHAFTS, side of the square} & = \frac{1}{5}\sqrt[3]{Pa}\end{array}\right\} \ldots (203)$$

The dimensions given by these formulas are unnecessarily large for a steady strain, but shafts are frequently subjected to sudden strains, amounting sometimes to a shock, and in these cases the results are none too large.

Practical formulas may also be established on the condition that the *total angle of torsion* shall not exceed a certain amount. Making $G = \frac{3}{8}E$, and solving (200) in reference to r, and we have for cylindrical shafts,

$$r = \sqrt[4]{\frac{16\,Pal}{3\pi E\alpha}},$$

and similarly for square shafts,

$$b = \sqrt[4]{\frac{16\,Pal}{E\alpha}}$$

---

in both cases. In this case the origin being at the centre of the square, we have $\int x^2 dA = \int y^2 dA \therefore Ip = 2\int y^2 dA = 2 \times \frac{1}{12} b^4$ (see Eq. (51)).

* *Résumé des Leçons*, Navier. Paris, 1856, pp. 193-203, and p. 507.

In these expressions P should not be so great as to impair the elasticity,—say for a steady strain P should not exceed the values given by Equation (203).

If α° is given in degrees, it is reduced to arc by multiplying it by $\frac{\pi}{180}$ so that $\alpha = \frac{\pi}{180}\alpha^\circ$; hence the preceding equations become: for cylindrical iron shafts,

$$r = 3.14\sqrt[4]{\frac{Pal}{E\alpha^\circ}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (204)$$

and for square iron shafts,

$$d = 5.51\sqrt[4]{\frac{Pal}{E\alpha^\circ}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (205).$$

Examples.—1. A round iron shaft 15 feet long, is acted upon by a weight $P = 2{,}000$ lbs. applied at the circumference of a wheel which is on the shaft, the diameter of the wheel being 2 feet; what must be the diameter of the shaft so that the total angle of torsion shall be 2 degrees?

If the shaft is cast-iron $E = 16{,}000{,}000$, and

$$2r = d = 6.28\sqrt[4]{\frac{2000 \times 12 \times 15 \times 12}{2 \times 16{,}000{,}000}} = 3.69 \text{ inches.}$$

2. A round wooden shaft, whose length is 8 feet, is attached to a wheel whose diameter is 8 feet. A force of 200 lbs. is applied at the circumference of the wheel, what must be the diameter of the shaft so that the total angle of torsion shall not exceed 2 degrees?

$$2r = d = 6.28\sqrt[4]{\frac{200 \times 4 \times 12 \times 8 \times 12}{2 \times 2{,}000{,}000}} = 4.35 \text{ inches.}$$

175*a*. RESULTS OF WERTHEIM'S EXPERIMENTS.—A few years since M. G. Wertheim presented to the French *Académie des Sciences* an exhaustive paper upon the subject of torsion, the substance of which was published in the *Annales de Chimie et de Physique*, Vol. XXIII., 1st Series, and Vols. XL. and L., 3d Series. These articles would make a volume by themselves, and hence we will content ourselves at this time with presenting his

CONCLUSIONS.

When a body of three dimensions is subject to torsion the following facts are observed:—

1st. The torsion angle will consist of two parts, one temporary, the other permanent; the latter augments continually, though not regularly.

2d. The temporary displacements augment more and more rapidly than the moments of the applied couples, and the increase of the mean angle, which in hard bodies continues until rupture, in soft bodies continues only to the point where the body commences to suffer rapid and continuous deformation.

3d. The temporary angles are not rigorously proportional to the length, and, all else being equal, the disproportionality increases in measure as the bar becomes shorter.

4th. In all homogeneous bodies, torsion caused a *diminution of the volume*, which is proportional to the length and square of the angle of torsion, and each point of the body, instead of describing an arc of a circle, follows the arc of a spiral. The condensation of the body increases from the centre to the circumference.

5th. In bodies with three axes of elasticity, the change of volume and resistance to torsion are functions of the free axes, and the relation between them may be such that the volume will augment.

6th. Circular or turning vibrations of great amplitude are difficult to produce, and as small angles of torsion only are used, the preceding conclusions apply to this case.

7th. Rupture produced by torsion usually takes place at the middle of the length of the prism; it commences at the dangerous points, and operates by slipping in hard bodies and by elongating in soft ones.

8th. With regard to the influence of the figure and absolute dimensions of the transverse sections of the bodies, we derive the following conclusions:—

9th. In homogeneous circular cylinders the diminution of the volume is equal to the original volume multiplied by the product of the square of the radius, and the angle of torsion for a unit of length (the angle being always very small). Further, under torsion the radius of the cylinder equals the primitive radius multiplied by the sine of the angle of inclination of the helicoidal fibres. This last gives a means of calculating the diminution of volume. But in reality the twisted cylinder takes the form of two frustra of cones joined at the smaller bases; and although this does not sensibly affect the theoretical results for long cylinders, yet it deprives our formulas of all their value in ordinary practical cases.

CHAPTER IX.

DISTORTIONS.

176. ANY CHANGE OF FORM OF A SOLID DUE TO FOREIGN FORCES IS A DISTORTION.—Several of these have been considered, separately and singly, in the preceding chapters, such as extension, compression, bending, torsion, and transverse shearing, but we shall find that in all cases one of these distortions is accompanied by some other one. In all elastic bodies the particles move more or less freely under the action of the straining forces.

The phenomenon of elasticity is nothing more than the action of the attractive and repulsive forces of the molecules of a body upon each other. When a force is applied to a body, its effect is transmitted from particle to particle by the internal forces, until it meets and is held in equilibrium by a force applied at some other part.

The Mathematical Theory of Elasticity is considered in three parts, the relation of stresses, the relation of strains, and the relations of stresses to strains.* We shall here consider only such principles as pertain immediately to the problems under consideration.

177. MEASURE OF SLIPPING.—If the section bd be forced into the position qf by the slipping (transverse shearing) of bd upon ac, the amount of the movement per unit of length will be measured by the angle baq, which for small displacements will be measured by the tangent of the angle.

Let g be the tangent of $baq = \frac{bq}{ab} = bq$ when ab is unity.

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* M. Lamé's *Leçons sur la Théorie Mathématique de l'Elasticité des Corps Solides*. Paris, 1852. *Résistance des Corps Solides*, par Navier. Troisième édition, 1864.

The resistance to this shearing will evidently vary as g, and also as the *elastic* resistance of the material, and if the resistance be evenly distributed over the transverse section it will also vary as this section.

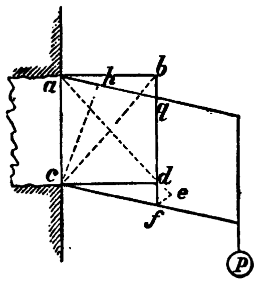

FIG. 109.

Let $P =$ the tangential force, that is, the force which acts in the plane bd,

$E_s =$ the coefficient of transverse elasticity,

$A =$ the area of the transverse section; then

$$P = E_s Ag \text{.......................} (206)$$

If $A = 1$, we have $p = E_s g$, which is the *intensity* of the stress.

When flexure is involved, we shall find that the shearing stress is not evenly distributed over the section. It is evenly distributed when bd is consecutive to ac, or when the area is small it may be considered uniform.

Letting fall the perpendicular ch from c upon aq, and we have

$$\frac{ah}{ch} = \frac{bq}{ab} = g;$$

hence, the transverse slipping in an amorphous body is accompanied by an equal longitudinal one, for we consider that the effect is the same as if ab had slipped over cd, an amount equal to ah.

Produce the diagonal ad and describe an arc fe which shall pass through f, having the centre at a, then will de be the elongation (or dilation) of the diagonal ad; and in a similar way we may find the contraction of the diagonal cb.

If $i =$ the dilation per unit of length, we have

$$i = \frac{de}{ad}$$

By similarity of the triangles *def* and *acd* we have

$$i = \frac{de}{ad} = \frac{\frac{cd}{ad}\,df}{\sqrt{ac^2 + cd^2}} = \frac{\frac{cd}{ad}\,\frac{df}{cd}}{\sqrt{1 + \frac{cd^2}{ac^2}}} = \frac{\frac{cd}{ad}}{\sqrt{1 + \frac{cd^2}{ac^2}}}g.$$

which is a maximum when $cd = ac$, for which case we have

$$i = \tfrac{1}{2}g \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (207)$$

or the maximum dilation is one-half the slipping. Similarly, the maximum contraction of *cd* takes place when it is the diagonal of a square.

We see that if we have two equal stresses in opposite senses, one a pull along *ad*, and the other a push along *cb*, whose directions make a right angle between them, the resulting distortion is equivalent to one-half of a simple shear of the same intensity on a plane at 45 degrees with either of the others.

Limit of the Slipping.

If $R' =$ the elastic limit of the strain;
$E =$ the coefficient of elasticity; and
$i' =$ the elongation produced by R', we have

$$i' = \frac{R'}{E} = \tfrac{1}{2}g \therefore g = \frac{2R'}{E}.$$

178. Relation between Longitudinal and Lateral Stresses.—When a body is subjected to a pull there is a lateral contraction, as shown in Fig. 116. The relation between these stresses, for bodies which are not homogeneous, is complex, but it is one of the questions which is considered in the *Mathematical Theory of Elasticity.* But for a solid whose elasticity is the same in all directions—called an isotropic body—the relations are comparatively simple. First consider the case in which the straining force acts in any direction.

Let p = a stress acting in any direction due to any cause,

p_{xx}* = the component of this pressure upon a unit of section resolved normally to a plane which is perpendicular to x;

p_{yy} = the normal component on a unit of section which is normal to y;

p_{zz} = similar component in regard to z;

A = the coefficient of direct or longitudinal elasticity, which expresses the relation between the longitudinal strains and the normal stresses;

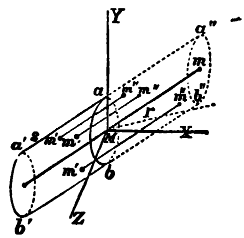

FIG. 110.

B = the coefficient of lateral elasticity, which expresses the relation between the longitudinal strain (a push or pull) and the stresses at right angles to the strains. It expresses the resistance to lateral contraction,

δ_x, δ_y, δ_z = the elongations in the directions of the axes x, y and z for a unit of length.

We then have, when the body is perfectly amorphous, or isotropic,†

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* This notation was first used by Cauchy and Corioles in discussions upon the Theory of Elasticity. The first sub-letter indicates the normal to the plane, and the second one the direction of action in that plane. Thus p_{xy} indicates a pressure upon a unit of area which is perpendicular to x, and in a direction parallel to y.

† Let $\overline{\Delta a}$ be an elementary section,
$r = Mm$ = the distance of any molecule, m to the right of M,
R = the force exerted by one molecule upon another at the distance and in the direction of r,
n the number of molecules contained in a unit of the body in the vicinity of M.

Take the origin of coördinates at M, x being taken perpendicular to the section $\overline{\Delta a}$, z vertical, and y perpendicular to $x\,z$.

The total action of all the molecules which are distant $Mm = r$ of the molecules on the right of M upon those at the left, is the same as if the whole mass of the cylinder at the left, whose length is r, were concentrated in the section at M acting upon the lamina at m; which is the same as if it were concentrated in the point M, and the lamina in the molecule m. We have

$$p_{xx} = A\delta_x + B(\delta_y + \delta_z)$$
$$p_{yy} = A\delta_y + B(\delta_z + \delta_x) \qquad (208)$$
$$p_{zz} = A\delta_z + B(\delta_x + \delta_y)$$

Suppose that the strain is parallel to x, then $p_{yy}=0$ and $p_{zz}=0$.

$\overline{\Delta a} \,.\, x =$ the volume of the cylinder,

$n\,\overline{\Delta a} \,.\, x =$ the number of molecules,

$Rn\,\overline{\Delta a} \,.\, x =$ the sum of all the actions parallel to r, between Mm,

$Rn\,\overline{\Delta a} \,.\, x\frac{x}{r} =$ the resultant normal to the section.

If we consider r as variable, and for each new value of r we substitute a proper corresponding value of R (which might be called R_1, R_2, R_3, etc.), we shall have a series of corresponding expressions all of which will have the same form as that given above, hence we have for all the forces which cross the section $\overline{\Delta a}$

$$\overline{\Delta a}\; SR\frac{x^2}{r}$$

in which S applies to all values of $n\,R\frac{x^2}{r}$ from zero to r, when the expression does not reduce to an insensible quantity on account of the rapid decrease of its value as r increases. The relation between R and r is not known, but we may assume that the resistance offered by the elastic forces above those which in the natural state are in equilibrium, when disturbed by an extraneous force, is proportional to the small increase of distance, dr, as we found in Chapter I. Let R_1 be the derivative of R in respect to r, then will the stress on a unit be

$$SR_1 dr\, x, \text{ and}$$

$$p_{xx} = SR_1 dr\,\frac{x^2}{r}$$

will be the resolved component of the stress.

Let δ_x, δ_y and δ_z be as given in the text, then

$\delta_x\frac{x}{r}x =$ the projection on r of the x component of the elongation, and similarly for y and z;

hence, neglecting all differences above the first, we have

$$dr = \delta_x\frac{x^2}{r} + \delta_y\frac{y^2}{r} + \delta_z\frac{z^2}{r}$$

hence

$$p_{xx} = \delta_x SR_1\frac{x^4}{r^2} + \delta_y SR_1\frac{x^2y^2}{r^2} + \delta_z SR_1\frac{y^2z^2}{r^2}$$

Similarly

$$p_{yy} = \delta_y SR_1\frac{y^4}{r^2} + \delta_z SR_1\frac{y^2z^2}{r^2} + \delta_x SR_1\frac{y^2x^2}{r^2}$$

Also the lateral compressions will be the same in the directions of y and z, and hence $\delta_y = \delta_z$. For this particular case let $\delta_x = \dot{i}_1$, $\delta_y = \delta_z = \dot{i}_2$, and we have

$$p_{xx} = A\dot{i}_1 + 2B\dot{i}_2 \ldots\ldots\ldots\ldots (208a)$$

$$0 = A\dot{i}_2 + B\dot{i}_2 + B\dot{i}_1 \ldots\ldots\ldots (209)$$

By elimination we find

$$\dot{i}_1 = \frac{A + B}{A^2 + AB - 2B^2} p_{xx}$$

$$\dot{i}_2 = \frac{-B}{A^2 + AB - 2B^2} p_{xx}$$

Hence $\dot{i}_2$ is negative compared with $\dot{i}_1$, as it should be, since a longitudinal pull produces a lateral compression.

The value of $\dot{i}_1$ when p_{xx} is unity is called the coefficient of direct pliability, and $\dot{i}_2$ the coefficient of lateral pliability.

Returning to the equations, and we find

$$A = \frac{\dot{i}_1 + \dot{i}_2}{\dot{i}_1^2 + \dot{i}_1\dot{i}_2 - 2\dot{i}_2^2} p_{xx};$$

$$B = \frac{-\dot{i}_2}{\dot{i}_1^2 + \dot{i}_1\dot{i}_2 - 2\dot{i}_2^2} p_{xx};$$

Or, since $\dot{i}_2$ is negative, we have for the numerical values of A and B when p_{xx} is unity, and $\dot{i}_1$ and $\dot{i}_2$ are both used as positive numbers;

$$A = \frac{\dot{i}_1 - \dot{i}_2}{\dot{i}_1^2 - \dot{i}_1\dot{i}_2 - 2\dot{i}_2^2}$$

$$p_{zz} = \delta_z SR_1 \frac{z^4}{r^2} + \delta_x SR_1 \frac{z^2x^2}{r^2} + \delta_y SR_1 \frac{z^2y^2}{r^2}$$

But on account of the isotropic character of the solid, the expressions which are similar will have the same value, hence

$$SR_1 \frac{x^4}{r^2} = SR_1 \frac{y^4}{r^2} = SR_1 \frac{z^4}{r^2} = A \text{ (say)},$$

$$SR_1 \frac{y^2z^2}{r^2} = SR_1 \frac{y^2x^2}{r^2} = SR_1 \frac{z^2x^2}{r^2} = B;$$

which reduces the preceding equations to those in the text.

It is more common in the investigations in molecular mechanics to prove at once a relation between A and B. The preceding is a *special* solution.

$$B = \frac{i_2}{i^2_1 - i_1 i_2 - 2i^2_2}$$

To find the relation between A and B requires a series of experiments, or a further consideration of molecular actions. Since the solid is isotropic, we shall here assume that the resistance to a shearing stress is the same in all directions, and since the lateral movement only takes place by shearing, we will here *assume* that the coefficient of lateral elasticity is the same as that for transverse shearing, or torsional shearing, or longitudinal shearing.

Generally let $C =$ the coefficient of transverse elasticity; the particular values of which will be for an isotrope,

$$C = B = E_s = \frac{P}{gA} \text{ (see Eq. 206)} = G \text{ (see Torsion).}$$

We have (as shown below),

$$A - B = 2C = \frac{1}{i_1 - i_2} p_{xx} = 2B$$

$$= \frac{1}{i_1 + i_2} p_{xx} \text{ numerically;}$$

$$\therefore C = \tfrac{1}{2}(A - B) \quad (210)$$

To prove this, take the case of two equal stresses acting at right angles with each other, in which one is a pull, and the other a push; the former being parallel to xx, and the latter parallel to yy. Since the body is isotropic, the contraction in the direction of z produced by p_{xx} will equal the expansion caused by p_{yy}, and hence δ_z, Eq. (208) will be zero, and $p_{zz} = 0$; hence the third of Eq. (208) gives $\delta_x = -\delta_y$, and Eqs. (208) become

$$p_{xx} = (A - B)\delta_x$$

$$p_{yy} = (-A + B)\delta_x = -p_{xx}$$

The intensity of the transverse shearing is, Eq. (206), $p = E_s g = Cg$ which for this case is, Eq. (207)

$$p = p_{xx} (\text{or } -p_{yy}) = C.2i = 2C\delta_x$$

hence $A - B = 2C$.

Assuming, as above stated, that the lateral elasticity for an isotrope is the same as the transverse, and we have

$$A - B = 2C = 2B$$
$$\therefore A = 3B$$

which in the second of Eqs. (209) gives

$$i_2 = -\tfrac{1}{4} i_1;$$

that is, the lateral contraction of an isotropic solid under the action of a direct stress is $\frac{1}{4}$ as much per unit as the longitudinal extension for the same unit.

If b = the breadth,
d = the depth,
l = the length of a prism,
V = the volume before elongation,
V_1 = " after " , and
P = the pulling force.

Then the elongation is

$$\lambda = \frac{Pl}{EK} \text{ (Eq. (1))}$$

$$\text{and } i_1 = \frac{\lambda}{l}$$

We also have

$$V = bdl$$
$$\therefore V_1 = b(1 - i_2)\, d(1 - i_3)\, l(1 + i_1)$$
$$= bdl(1 + i_1 - i_2 - i_3) \text{ nearly.}$$

If $i_2 = i_3$ exceeds $\frac{1}{2}i$, the expression becomes negative, or there would be a diminution of the volume, which is absurd; hence this may be considered a superior limit of the values of i_2 and i_3. If $i_2 = i_3 = \frac{1}{4}i_1$, we have

$$V_1 = bdl(1 + \tfrac{1}{2}i_1) = bd(l + \tfrac{1}{2}\lambda)$$

This solution shows, that on the convex side of a bent beam there will be a lateral contraction, and on the concave side there will be a lateral expansion. The elongation per unit of length for a rectangular section whose breadth is b, and depth d, is, Eq. (45),

$$\lambda = \frac{d}{2\rho}$$

and the lateral contraction per unit will be

$$\tfrac{1}{4}\lambda = \frac{d}{8\rho}$$

and the expansion on the opposite side will be the same amount. The total contraction will be

$$\frac{bd}{8\rho}$$

By this means we find that the lateral sides prolonged after flexure will meet at a point

$$4\rho$$

from the neutral axis opposite to the convex side. The sections which were rectangular before flexure become trapezoids after flexure.

Substitute the values of A and i_2 in Eq. (208a), and we find

$$p_{xx} = \frac{5}{2} B\, i_1 = \frac{5}{2}\, G i_1$$

But $\frac{p_{xx}}{i_1} = E$, the coefficient of longitudinal elasticity,

$$\therefore G = \frac{2}{5} E \ldots\ldots\ldots\ldots\ldots\ldots (211)$$

or the coefficient of elasticity for slipping (whether transverse, lateral, longitudinal, or torsional), in an isotropic solid is $\frac{2}{5}$ of the coefficient of longitudinal elasticity.

$$\text{But } G = B = \tfrac{1}{3} A;$$

$$\therefore E = \frac{5}{6} A,$$

that is, the coefficient of longitudinal elasticity is $\frac{5}{6}$ of that for the direct normal stress.

It will be seen from the equations, that the quantity E is a result of all the distortions, being the elongation which results from the yielding both longitudinally and laterally.

Returning again to Eqs. (208) and we have

$$\tfrac{1}{3}(p_{xx} + p_{yy} + p_{zz}) = \tfrac{1}{3}(A + 2B)(\delta_x + \delta_y + \delta_z)$$

The coefficient $\frac{1}{3}(A + 2B)$ expresses the relation between the mean direct stress and the cubic strains, and is called the *cubic elasticity*, or *elasticity of volume.*

The coefficient of transverse pliability is $\frac{1}{C} = c = 2\,(i_1 + i_2)$ (numerically).

The coefficient of cubic compressibility is

$$D = \frac{3}{A + 2B} = 3\,(i_1 - 2i_2)\ \text{(numerically)}.$$

The following are the coefficients for Crystal, as deduced from the experiments of M. Wertheim.—(*Annales de Chemie*, 3d Series, vol. xxiii.)

A	8,522,600
B	4,204,400
C	2,159,100
$\frac{1}{D}$	5,643,800
$\frac{1}{i_1}$	5,747,000
i_1	0.0000001740
i_2	0.0000000575
c	0.0000004631
D	0.0000001772

179. SHEARING STRAINS IN A BEAM WHICH IS BENT BY TRANSVERSE STRESSES.

In the discussions of the problems of flexure, the longitudinal elements were treated as if they produced no action upon each other, and were simply subjected to the laws of extension and compression.

If we conceive that the beam is composed of thin horizontal laminæ, and each had an uniform strain from one end to the other, there would be no slipping between the adjacent elements. This case is realized when the neutral axis is the arc of a circle, as when a beam is supported at its ends and is loaded with equal weights placed at equal distances from the supports. But in all other cases there is necessarily a slipping.

In Fig. 111, the element DC is not subjected to any tensile strain at C, but from that point the strain increases to D, where it will be a maximum. The strain can be unequal only by

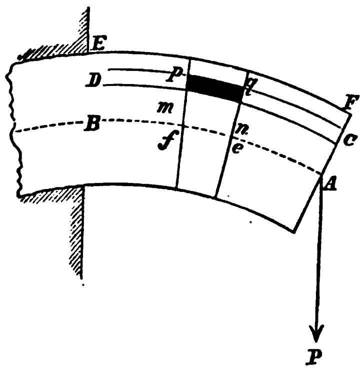

FIG. 111.

some adjacent element or elements taking off (so to speak) some of the pulling force, and the element must slide upon the adjacent one.

The moment of P in reference to e is Px, and in reference to f it is $P(x + dx)$, and the difference between these, or Pdx, is the moment of the shearing force at f in reference to e, and hence the shearing force is P, as given in Article 93. It remains to determine the law of distribution of this stress.

The strain on a unit of section qn is, according to Eq. (46),

$$p = \frac{E}{\rho} y$$

in which substitute the value of $\frac{E}{\rho}$ from Eq. (49), and we have

$$p = \frac{y}{I} \Sigma Px. \qquad (212)$$

and for a width mm' (Fig. 112) $= U$ the strain is

$$pU = \frac{y}{I} U \Sigma Px$$

and for a depth $qn = dy$ it is

$$pUdy = \frac{\Sigma Px}{I} yUdy.$$

The strain at pm is found in the same way, and the difference between the two is the excess of strain at m over that at n.

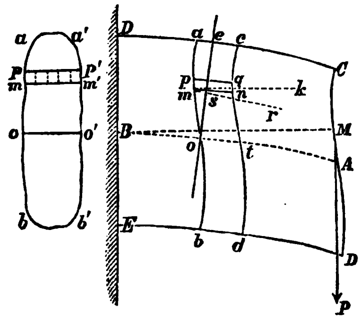

FIG. 112.

It is evidently the differential of the preceding expression in which p and ΣPx are the only variables (the beam being prismatic).

$$\therefore dp\,U\,dy = \frac{d\Sigma Px}{I}\,Uydy = \frac{\Sigma P.\,dx}{I}\,Uydy$$

(See Article 93 in regard to the last reduction.)

This is the value of the longitudinal shearing along $mn = dx$ due to the element mq. But the shearing due to all the elements between c and n will be

$$\frac{\Sigma Pdx}{I}\int_y^{d_1} Uydy,$$

(d_1 being the distance of the extreme fibre from the axis). This is the expression for the force which tends to move the volume $acnm$ along the line mn. This divided by the area $mn\,m'm = Udx$ gives the shearing strain on a unit of section, or its intensity, which is (say Gy, as stated on the next page.)

$$Gy = \frac{\Sigma P}{IU}\int_y^{d_1} Uydy \qquad (213)$$

which at the centre becomes

$$Gy_0 = \frac{\Sigma P}{Ib}\int_0^{d_1} Uydy \qquad (214)$$

in which b is the breadth on the neutral axis.

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But it was shown in Article 177 that a longitudinal shearing is accompanied by an equal transverse shearing. Hence, Eq. (213) gives the *intensity* of the transverse shearing at a point whose ordinate is y.

If ΣP becomes zero, as it does in the case of a Couple, Eq. (213) reduces to zero, or there is no transverse shearing.

The part $\int_y^{d_1} Uydy$ is the statical moment of $aa'mm'$ in reference to the neutral axis.

We see from this that the transverse and longitudinal shearing is zero at the upper and lower sides, and increases towards the neutral axis, at which place it is a maximum. We may conceive of this condition by supposing that the beam is built up of successive layers, each succeeding one being free, but adding to the shearing of all the preceding ones between it and the neutral axis. Equation (213) is not exact except for rectangular cross sections, for when the cross section is elliptical or otherwise curved, the free surface is at variable distances from the neutral axis.

If mk is normal to mp, and
mr is parallel to ot, then
g = tangent of the angle kmr, the amount of slipping,
G = coefficient of shearing elasticity, = E_s for isotropes, and
Gg = the resistance to shearing per unit, as used above.

The mean intensity is

$$\frac{\Sigma P}{A}$$

Hence, the ratio of the maximum shearing is to the mean as

$$\frac{Gg_0 A}{\Sigma P} = \frac{A}{Ib}\int_0^{d_1} Uydy$$

which depends entirely upon the form of section.

If the cross-sections are rectangular, Equation (214) gives

$$Gg_0 = \frac{\Sigma P}{\frac{1}{12}bd^3.b}\int_0^{\frac{1}{2}d} bydy = \tfrac{3}{2}\,\frac{\Sigma P}{bd} \;\ldots\ldots (215)$$

which is the maximum *intensity* of the shearing, and is $\frac{3}{2}$ of the mean. The total shearing is

$$\tfrac{2}{3}\, Gg_0 bd = \Sigma P$$

If the beam be fixed at one end and loaded at the free end, $\Sigma P = P$, and hence the total force which at all points along the neutral axis tends to push the upper half of the beam along the lower half, and which should be resisted by the cohesion of the elements, will be per unit of length

$$Gg_0 b = \tfrac{3}{2}\frac{P}{d}.$$

For any portion $aa'mm'$ of the rectangle we have

$$Gg = \frac{\Sigma P}{\frac{1}{12}bd^3}\int_y^{\frac{1}{2}d} bydy = \frac{6\Sigma P}{bd^3}\left[\tfrac{1}{4}d^2 - y^2\right] \ldots (216)$$

$$\text{or } G\,(g_0 - g) = \frac{6\Sigma P}{bd^3}y^2 \ldots\ldots\ldots\ldots\ldots\ldots (216a)$$

that is, *in rectangular beams the difference between the shearing at the neutral axis and any other point above or below it, varies as the square of the ordinate.*

The total transverse shearing in the cross section is

$$\int_{-\frac{1}{2}d}^{\frac{1}{2}d}(Ggb)dy = \frac{6\Sigma P}{bd^3}\int_{-\frac{1}{2}d}^{\frac{1}{2}d}\left[\tfrac{1}{4}d^2 - y^2\right]dy,$$

which by reduction gives ΣP, and agrees with Article 93.

It now appears evident that when there is longitudinal shearing, the transverse sections which were originally plane, will not remain plane during flexure.

To find the Equation of the Curve *aob* for rectangular beams,

Erect *oe* normally to *ot* at *o*, and let

$y =$ the ordinate *os*, and
$x =$ the abscissa *ms*,
mr is parallel to *ot*,
mk is normal to *mp*, and
$g =$ tangent *kmr* $=$ the slipping.

$\frac{dx}{dy}$ = the inclination of mp to oe, $= g$ plus a small inclination which mn may have in regard to ot, which is produced by the lateral contraction due to the longitudinal extension.

The elongation per unit of length is

$$\frac{y}{\rho} \quad \text{(Eq. (45))}$$

and the contraction for an isotropic solid is $\frac{1}{4}\frac{y}{\rho}$, and hence the contraction between o and m is

$$\int_0^y \frac{1}{4}\frac{y}{\rho}dy = \frac{y^2}{8\rho} = \frac{y^2}{8}\frac{\Sigma Px}{EI}$$

The distance mo will be less than tn by an amount which will not differ sensibly from the differential of the preceding expression. Differentiating and dividing the result by dx gives for the tangent of the inclination

$$\frac{y^2 \Sigma P}{8EI}$$

Hence, for rectangular beams we have

$$\frac{dx}{dy} = g + \frac{\Sigma P}{8EI}y^2 = \frac{6\Sigma P}{Gbd^3}\left[\tfrac{1}{4}d^2 - y^2 + \frac{G}{4E}y^2\right]$$

Integrating, observing that $x = 0$ for $y = 0$, and we find

$$x = \frac{3\Sigma P}{4Gb}\left[\frac{2y}{d} - \left(1 - \frac{G}{4E}\right)\frac{8y^3}{3d^3}\right]$$

Each half of the curve aob is therefore a parabola of the third degree, and of the same order as the curve AB.

For the ordinate ae, at the upper surface, make $y = \frac{1}{2}d$.

$$\therefore ae = \frac{\Sigma P}{16Gb}\left[8 + \frac{G}{E}\right] = \frac{1}{2}\frac{\Sigma P}{Gb} \text{ nearly} \ldots\ldots\ldots (217)$$

We here have the peculiar result that the total effect of the ongitudinal shearing at the surface is independent of the depth and length.

If $G = \frac{2}{5} E$, we have

$$x = \frac{3}{4} \frac{\Sigma P}{G b d^3} \left[5 d^2 y - 6 y^3 \right] \ldots\ldots\ldots\ldots\ldots (217a)$$

If $\Sigma P = 0$, as in the case of a Couple, x will be zero for all values of y, and hence the section will be plane.

The curve is normal to the neutral axis, or parallel to oe, where $\frac{dx}{dy} = 0$, or at the point whose ordinate is

$$y = 0.52d,$$

or at a short distance above the upper surface. Were there no lateral contraction it would cut the upper surface normally.

The algebraic curve, if continued, will cut the axis of y at two points, one at $y = 0$, and the other at $\sqrt{\frac{5}{6}}\, d$, or at 0.91 d nearly.

Example.—If $\Sigma P = P = 2{,}000$ lbs., $G = 8{,}000{,}000$ lbs., $b = 1$ inch, $d = 12$ inches, required the ordinate ae.

Ans. $ae = \frac{1}{4000}$ inch, nearly.

It will be seen from the preceding equation, that all the transverse sections in prismatic beams which were originally parallel, will be of the same form when the beam is loaded at one point only; and that it will be modified at the different points of the beam if the value of the shearing stress, $\Sigma P = S_s$ varies.

180. INCREASED DEFLECTION DUE TO TRANSVERSE SHEARING (OR SLIPPING).

The slope of any element due to the transverse shearing, Eq. (213), is

$$\text{tang}\, kmr = g = \frac{\Sigma P}{GIU} \int_y^{d_1} Uy dy,$$

which must be added to the slope due to bending by flexure.

The increased deflection for a length l will be

$$\delta' = lg$$

if g is constant, but if it is variable,

$\delta' \quad \int_0^x g dx$; or by substituting the preceding value of g, we have

$$\delta' = \int_0^x \frac{\Sigma P dx}{GIU} \int_y^{d_1} U y dy = \frac{\Sigma P x}{GIU} \int_0^{d_1} U y dy \ldots (217b)$$

If the beam is rectangular, the limits are 0 and $\frac{1}{2}d$;

$$\therefore \delta' = \tfrac{3}{2} \frac{\Sigma P x}{Gbd};$$

hence the deflection varies directly as the moment of the bending forces and inversely as the area of the transverse section.

If the beam is fixed at one end and loaded at the free end,

$$\delta' = \tfrac{3}{2} \frac{Pl}{Gbd}$$

Hence the total deflection will be (Eq. (57))

$$\varDelta = \frac{3Pl}{bd} \left[\frac{4l^2}{3Eb^2} + \frac{1}{2Gb} \right]$$

If it be uniformly loaded

$$\delta' = \tfrac{3}{4} \frac{wl}{Gbd}$$

If it be supported at its ends and loaded at the middle, $\Sigma P x = \frac{1}{4} Pl$, hence

$$\delta' = \tfrac{3}{8} \frac{Pl}{Gbd},$$

and hence the total deflection will be (Eq. (73))

$$\varDelta = \frac{Pl^3}{4Ebd^3} + \tfrac{3}{8} \frac{Pl}{Gbd} \ldots\ldots\ldots\ldots\ldots\ldots (218)$$

Prof. W. A. Norton, of New Haven, Ct., detected the existence of the last term, involving the deflection due to transverse shearing * directly from experiment. He assumed that this stress is uniformly distributed over the transverse sections, and deduced an equation of the same form as the preceding, the only difference being in the value of the coefficient, G. Although the stress increases from the outer surfaces to the neutral axis, as we have seen in Equation (216), yet the resistance to

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* *Van Nostrand's Eclectic Engineering Magazine*, Vol. 3, p. 70, 1871.

deflection of rectangular beams varies according to the same law, as if it were evenly distributed.

Prof. Norton's experiments were made upon prismatic white pine sticks, and the mean of a large number of experiments gave

$$E = 1,427,965 \text{ lbs.}$$

$$\text{and } \frac{3}{8G} = 0.0000094 \text{ lbs.*}$$

$$\therefore G = 40,000 \text{ lbs., very nearly.}$$

This is only $\frac{1}{35}$ of the value of E. So small a value at first caused a doubt as to the applicability of the formula to fibrous beams. But the experiments of Chevandier and Wertheim, p. 17, give for white pine the coefficient of elasticity in the direction of the radius 97.7 kil. per sq. millimetres, or 135,950 lbs., and in the direction of the tangent to the layers, 40,680 lbs., from which we see that the value given above may be correct for the material used, and for the position in which it was used.

Prof. Norton also informed the author that there were discrepancies in the experiments which he was not able to account for at the time; but that, in the light of Chevandier and Wertheim's experiments, he was of the opinion that they were mainly due to the position of the layers in the specimens, as some might have been horizontal, others vertical, and still others inclined, when the experiments were made. Duleau's experiments, Article 173, gave for the coefficient of elasticity for torsion (perpendicular to the direction of the fibres) 615,472 lbs. For fibrous bodies there is no simple relation between the coefficients of elasticity in the different directions. For cast iron, wrought iron and steel it is generally assumed that the coefficient of shearing elasticity is $\frac{1}{3}$ that for longitudinal elasticity.

Equation (218) may be written

$$\Delta = \frac{P.l^3}{4Ebd^3}\left[1 + \tfrac{3}{2}\frac{Ed^2}{Gl^2}\right] \quad \text{.......(219)}$$

* $\frac{3}{8G}$ he represented by C in his equation.

from which we see that the shorter the beam compared with its depth, the greater is the deflection due to transverse elasticity compared with that due to the direct elasticity. In very short beams nearly the whole deflection may be due to shearing, while in long ones it may generally be neglected.

181. DEFLECTION DUE TO LONGITUDINAL SLIPPING.—If the lamina were free to slip upon each other, as we have before illustrated by a pile of boards, they would retain their original length, and the deflection would be much greater than if there were no slipping. If the elements were held together by cohesion, but had no longitudinal shearing elasticity, but had, as now, a direct longitudinal elasticity and a transverse shearing elasticity, Equation (218) would give the deflection for rectangular beams supported at their ends and loaded at the middle. If the sections remained plane and were forced past each other, as in Fig. 26, without bending by flexure, then the total deflection would be given by Formula (217*b*). But there is a longitudinal shearing *stress* at every point where there is a transverse shearing, and the elasticity of the material permits a corresponding longitudinal shearing *strain*, and hence there is slipping, and the longitudinal elements are independent of each other to just the extent of the slipping and no more. When the longitudinal shearing elasticity is the same as the transverse, Equation (217) shows the total effect in a cross section of the shearing in a rectangular beam.

Join o and a, Fig. 212, with a straight line, and conceive that the beam turns about o so as to produce an opening ae at a. This would cause a deflection which we call $d\delta$, and by similar triangles we have

$$ae : \tfrac{1}{2}d :: d\delta : x$$

$$\therefore d\delta = \frac{2\,ae}{d}x$$

$$= \frac{\Sigma P}{8Gbd}\left[8 + \frac{G}{E}\right]x = \frac{\Sigma P}{Gbd}x \text{ nearly.}$$

Although this is not the correct expression due to longitudinal slipping, yet we may safely assume that it is proportional to it. The longitudinal shearing elasticity may differ from the transverse. If then G_1 be the coefficient of longitudinal shearing elasticity, and r the ratio between the preceding expression and the true one, then the expression for the deflection due to this cause will be

$$\frac{\Sigma P}{rG_1bd}$$

But the total deflection will be the sum of the expressions which result by giving to x all possible values from $x = 0$ to $x =$ the length considered;

$$\therefore \delta = \frac{\Sigma P}{rG_1bd}\int xdx;$$

which for a rectangular beam fixed at one end and loaded at the free end, becomes

$$\delta = \frac{Pl^2}{2rG_1bd};$$

and if supported at its ends and loaded at the middle, $\Sigma P = \frac{1}{2}P$, and

$$\delta = \frac{Pl^2}{16rG_1bd}.$$

This added to Equation (218) gives for *the total deflection of a rectangular beam which is supported at its ends and loaded at the middle,*

$$\varDelta = \frac{Pl^3}{4Ebd^3} + \frac{Pl^2}{16rG_1bd} + \frac{3Pl}{8Gbd}$$

$$= \frac{Pl}{4bd}\left[\frac{l^2}{Ed^2} + \frac{l}{4rG_1} + \frac{3}{2G}\right] \ldots\ldots\ldots (219a)$$

NOTE.—As these pages are passing through the press, the author has received the following note from Professor Norton, which I am pleased to insert in this place, although I do not agree with his theoretical views. The fact, however, that his new formula represents the results of his experiments so accurately makes it worthy of serious consideration:

"I find that the entire series of experiments which I have made on the deflection of pine sticks, and iron and steel bars, loaded at the middle and resting on supports, are represented with great accuracy by the following formula:

$$\varDelta = C\frac{Pl^2}{bd} + \tfrac{2}{3}.\frac{Pl^3}{4Ebd^3}$$

in which l denotes the length, b the breadth, and d the depth of the rectangular stick or bar, P the load at its middle, E the coefficient of elasticity, and C a constant coefficient. The old formula, $\Delta = \frac{Pl^3}{4Ebd^3}$ involves two laws, which it appears from my experiments are very wide of the truth, unless the ratio of the length to the depth is large. These laws are, that the deflection is directly proportional to the cube of the length, and inversely proportional to the cube of the depth. The coefficient of elasticity, E, as determined with it, from any observed deflection, Δ, is ordinarily much too large. For example, it gives me for the wrought-iron bar I have used, as the value of E, 36,833,-000 lbs., and for the steel bar 37,066,000 lbs.

"The formula which I before obtained is

$$\Delta = C\frac{Pl}{bd} + \frac{Pl^3}{4Ebd^3}$$

This, with certain values of E and C, represented my experiments with white pine sticks much more satisfactorily than the old formula just referred to; but the experiments on the deflection of wrought-iron and steel bars have shown it to be faulty—whether regarded as an empirical formula, or from the theoretical point of view—though it gives results that approximate more nearly to the truth than the one in general use.

"In the new formula the first term represents the portion of the linear deflection resulting directly from the shearing stress, propagated from the middle of the bar to either point of support. This I had before conceived to be proportional to the length of the bar, but it appears, on careful consideration, that it is proportional to the square of the length. This may be seen by taking the case of a bar fastened at one end and loaded at the free end, and reflecting that when any one material section slips on the next one on the side towards the support, it must take down with it in this act all the bar between it and the free end, just as if this were an index extending out from the point considered. The molecular actions by which this is effected will undoubtedly give rise incidentally to small longitudinal strains, by which the relative positions of the molecules of the two contiguous sections will be somewhat disturbed. Accordingly, the linear deflection of the end of the bar, resulting from the shearing stress taking effect along the whole length of the bar, should be proportional to the square of the length. The same conclusion will obviously apply to the case of a bar loaded at its middle and resting on two supports. The effect of the shearing stress should also be inversely proportional to the area of the cross section, or bd, and directly proportional to $\frac{P}{2}$. We thus obtain, on theoretical grounds, the term $C'\frac{P}{2}\cdot\frac{l^2}{4}\cdot\frac{1}{bd}$. Now let G = coefficient of elastic resistance to transverse shearing stress, and let $m = \frac{E}{G}$, or $G = \frac{E}{m}$. The value of C^1 for the distance 1 is $\frac{1}{G}$, or $\frac{m}{E}$. The new term becomes, then, $\frac{m}{8E}\cdot\frac{Pl^2}{bd}$, for which we may take $C\frac{Pl^2}{bd}$.

"The other term in the formula, viz., $\frac{2}{3}\frac{Pl^3}{4Ebd^3}$ differs from the ordinary expression for the deflection due to the longitudinal strains on the fibres, in containing the factor $\frac{2}{3}$. With regard to this I will only state here that this is the expression for the deflection attendant on the longitudinal strains, which I deduced a year or two since, directly from the fundamental conception that these strains are the incidental effects of the shearing stress, without using the principle of the lever or of moments. I now find it to be verified by the results of my experiments. It may be well to add that this expression was not adopted, in my recent attempt to represent the experiments, until the ordinary expression $\frac{Pl^3}{4Ebd^3}$ was found to fail, giving values of E far too large.

"For the pine sticks the value of m, as derived from the values of C and E by the relation $C = \frac{m}{8E}$, falls between 2 and 4. It varies with the different sticks used, with the inclination of the layers to the horizontal. When the layers were horizontal, the value of m was 4; when they were vertical it was 2. In one of the sticks the inclination was nearly 45°, and the value of m came out 2.93. In the case of another stick the inclination, in one experiment, was about 25°, and in another 65°; and the values of m obtained are 2.18 and 3.55, the average of which is 2.86. The values of E obtained varied with the different sticks from 933,000 to 1,093,000 lbs.

"For the wrought-iron and steel bars (1 in. by $\frac{1}{2}$ in.) the value of m was a little less than 2 when the bar rested edgewise on its supports, and 4 when it rested flatwise; or the same as with pine sticks when their layers were vertical or horizontal. This is a remarkable result, since it indicates that the bars were made up of laminæ parallel to the breadth, with separating spaces of weaker molecular forces, called into action by vertical displacements. This condition of things is no doubt attributable to the operation of rolling, to which the bars have been subjected. This, I conceive, from theoretical considerations, ought to have had the effect to weaken the effective molecular forces in the vertical direction, and augment them in the horizontal direction. The value of E, for the iron bar, was found to be 25,220,000 lbs., and for the steel bar 25,333,000 lbs."

182. DIRECTION OF MAXIMUM AND MINIMUM STRAINS *at any point of the longitudinal section of a beam.*

Let $ABCD$ be an element,

p = the pull or push *per unit* of section AD, $= \frac{\Sigma Px}{I}y$ Eq. (212).

X = the shearing *per unit* on the surface AB,

$$= \frac{\Sigma P}{IU}\int_y^{d_1} Uydy, \text{ Eq. (213)}$$

which for rectangular beams is $\frac{6\Sigma P}{bd^3}[\frac{1}{4}d^2 - y^2]$, Eq. (216),

$Z =$ the transverse shearing *per unit* on AD, $= X$,
$\beta =$ the variable angle ABD,
$N =$ the normal component of the stresses *per unit* on the plane BD, and
$Ss =$ the shearing stress *per unit* along BD.

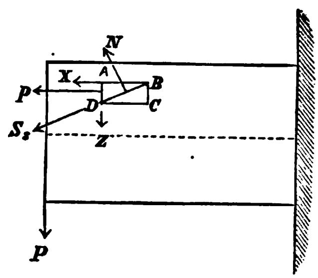

FIG. 113.

For the sake of simplicity, consider a rectangular beam, which is fixed at one end and loaded with a weight P at the free end. Then $\Sigma P = P$, and we have

$$p = \frac{12Pxy}{bd^3}, \quad \text{................(220)}$$

$$\text{and } X = Z = \frac{6P}{bd^3}[(\tfrac{1}{2}d)^2 - y^2] \quad \text{................(221)}$$

We also have

$X.AB =$ the total shearing on AB,
$Z.AD = X.AD =$ the total shearing on AD,
$p.AD =$ the direct pull or push on AD,
$N.BD =$ the total normal component on BD, and
$Ss.BD =$ the total shearing pull or push on BD.

Resolving the forces normally and parallel to BD, and we have

$$Ss.BD = AB.\ X\cos\beta - AD.\ X\sin\beta + AD.p\cos\beta$$
$$N.BD = AB.X\sin\beta + AD.X\cos\beta + BD.p\sin\beta$$

$$\text{But } \frac{AB}{BD} = \cos\beta\text{; and } \frac{AD}{BD} = \sin\beta$$

$$\therefore Ss = X(\cos^2\beta - \sin^2\beta) + p\sin\beta\cos\beta = X\cos 2\beta + \tfrac{1}{2}p\sin 2\beta \quad \text{(222)}$$

$$N = 2X\sin\beta\cos\beta + p\sin^2\beta = 2X\sin 2\beta + \tfrac{1}{2}p(1 - \cos 2\beta) \quad \text{(223)}$$

Hence, Ss is a maximum for

$$\text{tang}2\beta = \frac{p}{2X} = \text{tang}2\beta_1 \text{ (say)} \quad \text{(224)}$$

$$\therefore \sin 2\beta_1 = \frac{p}{\sqrt{p^2 + 4X^2}}; \quad \cos 2\beta_1 = \frac{2X}{\sqrt{p^2 + 4X^2}}$$

hence the maximum of the shearing force is

$$Ss_m = \sqrt{(\tfrac{1}{2}p)^2 + X^2} \quad \text{(224a)}$$

Similarly, for N we have for a maximum or minimum

$$\text{tang}2\beta_2 = -\frac{2X}{p} \quad \text{(225)}$$

$$\therefore \text{tang}2\beta_1 \, \text{tang}2\beta_2 = -1$$
$$\text{or } 2\beta_1 = 90^\circ + 2\beta_2$$
$$\therefore \beta_1 = 45^\circ + \beta_2;$$

hence the lines of maximum shearing cut the lines of maximum direct stress at angles of 45 degrees.

$2\beta_2$ may have two values a and $180^\circ + a$ $\therefore \beta = \tfrac{1}{2}a$ or $90^\circ + \tfrac{1}{2}a$

$$\sin 2\beta_2 = \mp \frac{2X}{\sqrt{p^2 + 4X^2}}; \quad \cos 2\beta_2 = \pm \frac{p}{\sqrt{p^2 + 4X^2}},$$

the upper signs correspond to a minimum and the lower to a maximum.

The maximum value of N is

$$N_m = \tfrac{1}{2}p + \sqrt{(\tfrac{1}{2}p)^2 + X^2}$$

and the minimum value is

$$N_0 = \tfrac{1}{2}p - \sqrt{(\tfrac{1}{2}p)^2 + X^2}$$

On the compressive side of the beam we have

$$Ss = X(\cos^2\beta - \sin^2\beta) - p\sin\beta\cos\beta$$
$$N = 2X\sin\beta\cos\beta - p\sin^2\beta$$

DISCUSSION.

On the neutral axis $p = 0$

$\therefore \text{tang}2\beta_1 = 0$ and $2\beta_1 = 0$ or $180° \therefore \beta_1 = 0$ or $90°$

$\text{tang}2\beta_2 = \infty$; $2\beta_2 = 90°$ or $270° \therefore \beta_2 = 45°$ or $135°$

and $Ss_m = X$ and $N_m = X$ and $N_o = -X$

This shows that the intensity of the shearing stress on the neutral axis is a maximum along the axis, and is of the same value as at right angles to it and equals X. The equality was shown in Art. 177. The maximum direct stress is normal to a section which is inclined 135 degrees to the axis, and the minimum at an angle of 45 degrees. Its value per unit is $N_m = X$. This may be shown directly, for it is evidently the resultant of two rectangular shearing forces each equal to X, and hence is $\sqrt{2}X$; but the area is $\sqrt{2}$ times the horizontal unit; hence the stress per unit will be $\sqrt{2}X \div \sqrt{2} = X$, as given above.

At the outer elements $X = 0$

$\therefore \text{tang}2\beta_1 = \infty$; $2\beta_1 = 90°$ and $\beta_1 = 45°$

$\text{tang}2\beta_2 = -0$; $2\beta_2 = 180°$ or $0°$ and $\beta_2 = 90°$ or $0°$

$Ss_m = \frac{1}{2}p$

$N_m = p$ and $N_o = 0$

That is, the maximum stress is normal to a section which is perpendicular to the neutral axis; in other words, it is parallel to the axis and equals the pulling stress, as it should. The minimum value is zero in a direction normal to the surface, and the maximum shearing stress is along a section which is inclined 45 degrees to the axis, and its intensity is $\frac{1}{2}p$, which agrees with Article 177.

For any point we have from Eqs. (220), (221), (224) and (225);

$$\text{tang}2\beta_1 = \frac{xy}{\frac{1}{4}d^2 - y^2}$$

$$\text{tang}2\beta_2 = -\frac{\frac{1}{4}d^2 - y^2}{xy} \qquad \text{.........(226)}$$

In the last equation take successive values for x (as $x = \frac{1}{8}l$, $\frac{2}{8}l$, $\frac{3}{8}l$, etc.), and for each value substitute values of y (such as $\frac{1}{5}$ of $\frac{1}{2}d$, $\frac{2}{5}$ of $\frac{1}{2}d$, etc.), and determine the corresponding values of β_2. Lay off the computed angles at the points whose coördinates are thus assumed. The lines of maximum direct stress will be normal to the lines thus constructed. But since the angles for a maximum and minimum differ by 90 degrees, the inclination of the lines so constructed will correspond in direction with (say) the maximum, and the normal to it with the minimum stress. By determining a sufficient number of points a network of lines may be drawn, as in Fig. 114, which represent

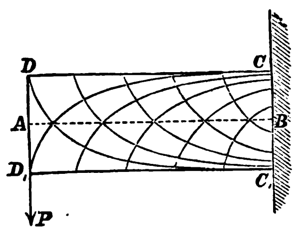

FIG. 114.

the direction of the lines of maximum and minimum stress, those concave downwards corresponding to tension, and those concave upwards to compression.

The parts more nearly horizontal correspond to the maximum, and the steeper parts to a minimum. They cross the neutral axis at angles of 45 degrees, and each other at all points at right angles, and the axes of minimum stress cut the surfaces at right angles, and the axes of maximum stress are parallel to it at the surface.

In a similar manner lines of maximum shear may be drawn.

To find the equation of one of these curves, we have (the axis of the stress being normal to the elementary section)

$$\frac{dy}{dx} = \cot\beta_2 \text{ or } \frac{dx}{dy} = \text{tang}\beta_2$$

$$\text{But tang}2\beta_2 = \frac{1 - \text{tang}^2\beta_2}{2\text{tang}\beta_2} = -\frac{2X}{p}$$

$$\therefore \frac{dx}{dy} = \text{tang}\beta_2 = \pm\frac{p}{2X} + \sqrt{1 + \frac{p^2}{4X^2}} = -\frac{N_0}{X} \text{ or } \frac{N_m}{X}$$

whence, by Equations (220) and (221)

$$dx = \left[\frac{xy}{\frac{1}{4}d^2 - y^2} + \sqrt{1 + \frac{x^2y^2}{(\frac{1}{4}d^2 - y^2)^2}}\right]dy$$

which is the differential equation of the curve, but I do not think that it can but integrated in finite terms.

Remark.—The analysis by which Poisson and others determined that the coefficient of lateral contraction is $\frac{1}{4}$ that of the longitudinal dilation, both per unit, has been criticised by Thomson and Tait. (See their *Natural Philosophy*, 1867, Vol. I., p. 521.) They give the following:—

Let n = the *rigidity*, which according to our notation $= C = \frac{1}{2}(A - B)$, and k = the *resistance to dilation* $= \frac{1}{D} = \frac{1}{3}(A + 2B)$; then the linear elongation, $i_1 = \left(\frac{1}{3n} + \frac{1}{9k}\right)P = \frac{A+B}{A^2+AB-2B^2}P$, and the linear contraction, $i_2 = \left(\frac{1}{6n} - \frac{1}{9k}\right)P = \frac{B}{A^2+AB-2B^2}P$, which are the values of i_1 and i_2 following Eq. (209) in the preceding text.

We have

$$\frac{i_2}{i_1} = \frac{B}{A+B};$$

in which, if the ratio is $\frac{1}{4}$, B will be $\frac{1}{3}A$. These substituted in Eq. (208a) give

$$E = \tfrac{5}{6}A = \tfrac{5}{2}B \text{ (or } \tfrac{5}{2}G),$$

as before found. That this result is approximately true for iron has been shown by the experiments of M. Wertheim.*

For ordinary glass and crystal he found 2.4 nearly for the ratio.

But there are some isotropic solids in which this is not the correct ratio, such as India rubber and elastic jellies. In such cases the value of B must be determined by experiment.

I assumed that $C = B = G$, because it is approximately true for those solids which are more commonly used by the engineer, and also because it greatly simplifies the investigation.

The problem of the distortion of a prism which is subjected to torsion has been thoroughly discussed by St. Venant. He determined the character of the sections which originally were plane and normal to the axis of torsion; also determined the correction which should be applied to Coulomb's formula; also compared his results with those of experiment, and deduced conclusions of great value to the engineer. This problem alone furnishes sufficient material to fill a volume.

It was unnecessary to introduce the letter C into the notation on page 220 and the following, since it is the same as E_s previously used; but I did so because it has been used by other writers, and I desired to show its relation to my notation.

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* *Comptes Rendus*, t. xxviii., p. 126.

CHAPTER X.

EFFECT OF LONG-CONTINUED STRAINS—OF OFT-REPEATED STRAINS, AND OF SHOCKS—REMARKS UPON THE CRYSTALLIZATION OF IRON.

EFFECT OF LONG-CONTINUED STRAINS.

183. GENERAL EFFECT.—The values of the coefficients of elasticity and the modulii of tenacity, crushing, and of rupture were determined from strains which were continued for a short time—generally only a few minutes—or until equilibrium was apparently established; and yet it is well known that if the strain is severe, the distortion, whether for extension, compression, or bending, will increase for a long time; and as for rupture, it always takes time to break a piece, however suddenly rupture may be produced. By sudden rupture we only mean that it is produced in a very short time.

The *increased* elongation due to a prolonged duration of the strain beyond a few minutes will affect the coefficient of elasticity but very slightly, for the strains which are used in determining it are always comparatively small, and the greater part of the effect is produced immediately after the stress is applied. If the distortion should go on indefinitely under the action of a constant load, no matter how slowly, the elasticity, and hence the coefficient, would be greatly modified by a very great duration of the stress; and at last rupture would take place. If the basis of this reasoning be well founded, we might reasonably fear the ultimate stability of all structures, and especially those in which there are members subjected to tension. But the continued stability of structures which have stood for centuries, teaches us, *practically at least*, that in all cases in which the strain is not too severe, equilibrium becomes established between the stresses and strains, and in such cases the piece will sustain the stress for an indefinitely long time.

184. HODGKINSON'S EXPERIMENTS.—The results of the experiments which are recorded in Article 45, page 52, show that in one case the compression increased with the duration of the strain for three-fourths of an hour. In the case of extension on another bar, as shown in Article VII., page 7, it appears that the same weight produced an increased elongation for nine hours; but during the last, or tenth hour, there was no increase over that at the end of the ninth hour.

In both these cases the strain was more than one-half that of the ultimate strength.

185. VICAT'S EXPERIMENTS.— M. Vicat took wrought-iron wire and subjected it to an uniform stress for thirty-three months. The elongations produced by the several weights were measured soon after the weights were applied, and total lengths determined from time to time during the thirty-three months. It was found for all but the first wire, as given in the following table, that the increased elongations after the first one were very nearly proportional to the duration of the stress. (*Annales de Chemie et Physique*, Vol. 54, 2d series.)

TABLE

Of the Results of M. Vicat's Experiments on Wrought-iron Wire.

Amount of Strain.		Increased Elongation after 33 months.
$\frac{1}{4}$ of its ultimate tensile strength.	Very soon after the weights were laid on the elongation of each piece was determined.	No additional increase.
$\frac{1}{3}$ of its ultimate tensile strength.		0.027 of an inch per foot.
$\frac{1}{2}$ of its ultimate tensile strength.		0.040 of an inch per foot.
$\frac{3}{4}$ of its ultimate tensile strength.		0.061 of an inch per foot.

186. FAIRBAIRN'S EXPERIMENTS.—Fairbairn made experiments upon several bars of iron, which were subjected to a transverse strain, the results of some of which are recorded in the following tables. (*See Cast and Wrought Iron, by Wm. Fairbairn*). The bars were four feet six inches between the supports, and weights were applied at the middle, and permitted to remain there several years, as indicated by the tables. The deflections were noted from time to time, and the results were recorded.

TABLE I.

In which the Weight Applied was 336 *pounds.*

TEMPERATURE.	Date of Observation.	Cold-blast—deflection in inches.	Hot-blast—deflection in inches.	Ratio of load to mean breaking weight.
	March 11, 1837....	1.270	1.461	Cold-blast,
78°	June 3, 1838.......	1.316	1.538	0.661 : 1
72°	July 5, 1839.......	1.305	1.533	Hot-blast,
61°	June 6, 1840.......	1.303	1.520	0.694 : 1
50°	November 22, 1841.	1.306	1.620	
58°	April 19, 1842......	1.308	1.620	
	Mean...........	1.301	1.548	

Previous to taking the observations in November and April the hot-blast bar had been disturbed.

In regard to this experiment Mr. Fairbairn remarks:—"The above experiments show a progressive increase in the deflections of the cold-blast bar during a period of five years of 0.031 of an inch, and of 0.087 of the hot-blast bar." The numerical results are found by comparing the first deflection with the mean of all the observed deflections. But an examination of the table shows that the greatest deflection, which was observed in both cases, was at the second observation, which was about a year and a quarter after the weight was applied, and during the next two years the *deflections decreased* 0.015 of an inch for the cold blast, and 0.018 of an inch for the hot-blast bar. After this the deflections appear to increase for the cold-blast bar 0.005 of an inch the next two years. Considering all the particulars of these experiments it does not seem just to conclude that the deflections would have gone on increasing indefinitely with a continuance of the load. Admitting that the small increase of deflections during the last two years are correct and not due to errors of observation, we see no reason why the deflections would not be as likely to decrease after a time as they were after the first year.

TABLE II.

In which the Bar was Loaded with 392 pounds.

TEMPERATURE.	Date of Observation.	Cold-blast—deflection in inches.	Hot-blast—deflection in inches.	Ratio of load to mean breaking weight.
	March 6, 1837......	1.684	1.715	
78°	June 23, 1838......	1.824	1.803	For cold-blast,
72°	July 5, 1839.......	1.824	1.798	0.771 : 1
61°	June 6, 1840.......	1.825	1.798	For hot-blast,
50°	November 22, 1841.	1.829	1.804	0.805 : 1
58°	April 19, 1842......	1.828	1.812	
	Mean...........	1.802	1.788	

Here we see a general increase in the deflections from year to year, but the changes are not entirely regular. The principal *increase* is during the first year. In the cold-blast there was a slight decrease of deflection during the last year, and in the cold-blast it was less at the third and fourth measurement than at the second.

TABLE III.

TEMPERATURE.	Date of Observation.	Cold-blast—deflection in inches.	Hot-blast—deflection in inches.	Ratio of permanent load to mean breaking weight.
	March 6, 1837......	1.410	Broke immediately with 448 pounds.	Cold-blast, 0.881 : 1
78°	June 23, 1838......	1.457		
72°	July 5, 1839.......	1.446		
61°	June 6, 1840.......	1.445		
50°	November 22, 1841.	1.449		
58°	April 19, 1842......	1.449		
	Mean...........	1.442		

We find from this table, as from Table I., that the maximum deflection was observed about a year and a quarter after the weight was applied, and that it decreased during the next two years, after which it slightly increased. The deflections were the same at the two last observations. These changes took place under the severe strain of more than four-fifths of the breaking weight. These experiments *indicate* that for a steady strain which is less than three-fourths of the ultimate strength of the bar, the deflection will not increase progressively until rupture takes place, but will be confined within small limits.

187. ROEBLING'S OBSERVATIONS.—The old Monongahela bridge in Pennsylvania, after thirty years of severe service, was removed to make place for a new structure. The iron which was taken from the old structure was carefully examined and tested by Mr. Roebling, and found to be in such good condition that it was introduced by him into the new bridge.*

He also found that the iron in another bridge over the Alle-

* Roebling's *Report on the Niagara Railroad Bridge*, 1860, p. 17; *Jour. Frank. Inst.*, 1860, Vol. LXX., p. 361.

ghany River was in good condition after forty-one years of service.

188. OFT-REPEATED STRAINS.—Nearly all kinds of structures are subjected to greater strains at certain times than at others, and some structures, as bridges and certain machines, are subject to almost constant changes in the strains. Loads are put on and removed, and the operation constantly repeated. The following experiments for determining the effect of a load which is placed upon a bar and then removed, and the operation of which was frequently repeated, were made by Wm. Fairbairn, in 1860.* The beam was supported at its ends, and the weight which produced the strain was raised and lowered by means of a crank and pitman, as in Fig. 115.

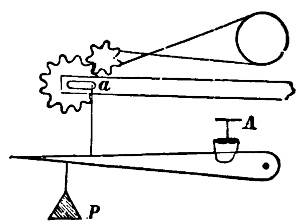

FIG. 115.

The gearing was connected with a water-wheel, which was kept in motion day and night, and the number of changes of the load were registered by an automatic counter. The beam was 20 feet clear span and 16 inches deep. The dimensions of the cross section were as follows:

Top—Plate, $4 \times \frac{1}{2}$ =	2.00	sq. inches.
Angle irons, $2 \times 2 \times \frac{5}{16}$ =	2.30	" "
Bottom—Plate, $4 \times \frac{1}{2}$ =	2.00	" "
Angle irons, $2 \times 2 \times \frac{3}{16}$ =	1.40	" "
Web—Plate, $15\frac{1}{4} \times \frac{1}{8}$ =	1.90	" "
Total	8.60	" "

Weight of beam, 1 cwt. 3 qrs. 3 lbs.
Probable breaking weight, 9.6 tons.

* *Civ. Eng. and Arch. Jour.*, Vol. XXIII., p. 257, and Vol. XXIV., p. 237.

First Experiment.—Beam loaded to ¼ the breaking weight:—

Total applied load................5,809 lbs.
Half the weight of the beam....... 434 "
Strain on the bottom flange........4.3 tons per sq. inch.
Margin of strength by Board of Trade........................3.4

TABLE

Of the Results of Experiments made upon a Beam which was Supported at its Ends, and a Weight repeatedly but gradually Applied at the Middle.

DATE.	No. of Changes.	Deflection at Centre of Beam.	DATE.	No. of Changes.	Deflection at Centre of Beam.
1860.			1860.		
March 21.........	......	0.17	April 13.........	268,328	0.17
22.........	10,540	0.18	14.........	281,210	0.17
23.........	15,610	0.16	17.........	321,015	0.17
24.........	27,840	....	20.	343,880	0.17
26.........	46,100	0.16	25.........	390,430	0.16
27..... ...	57,790	0.17	27.........	408,264	0.16
28.........	72,440	0.17	28.........	417,940	0.16
29.........	85,960	0.17	May 1.........	449,280	0.16
30.........	97,420	0.17	3	468,600	0.16
31 ...	112,810	0.17	5.........	489,769	0.16
April 2.........	144,350	0.16	7.........	512,181	0.16
4.........	165,710	0.18	9.........	536,355	0.16
7.........	202,890	0.17	11.........	560,529	0.16
10.........	235,811	0.17	14.........	596,790	0.16

At this point, after half a million of changes, the beam did not appear to be damaged. At first it took a permanent set of 0.01 of an inch, which did not appear to increase afterwards, and the mean deflection for the last changes were less than for the first. For the last seventeen days the deflection was uniform, but for the first seventeen days it was variable.

The moving load was now increased to one-third the breaking weight, = 7,406 lbs., with the following results:

DATE.	No. of Changes of Load.	Deflection in Inches.	DATE.	No. of Changes of Load.	Deflection in Inches.
1860.			1860.		
May 14.........	......	0.22	June 7.........	217,300	0.21
15.........	12,623	0.22	9.........	236,460	0.21
17.........	36,417	0.22	12.........	264,220	0.21
19.........	53,770	0.21	16.........	292,600	0.22
22.........	85,820	0.22	21.........	327,000	0.23
26.........	128,300	0.22	23.........	350,000	0.25
29.........	161,500	0.22	25.........	375,650	0.23
31.........	177,000	0.22	26.........	403,210	0.23
June 4.........	194,500	0.21			

The beam had now received 1,000,000 changes of the load, but it remained uninjured. The moving load was now increased to 10,050 lbs.—or one-half the breaking-weight—and it broke with 5,175 changes. The beam was then repaired by riveting a piece on the lower flange, so that the sectional area was the same as before, and the experiment was continued. One hundred and fifty-eight changes were made with a load equal to one-half the breaking weight; and the load was then reduced to two-fifths the breaking weight, and 25,900 changes made. Lastly, the load was reduced to one-third the breaking weight, with the following results:—

DATE.	No. of Changes of Load.	Deflection in Inches.	DATE.	No. of Changes of Load.	Deflection in Inches.
1860.			1860.		
August 13	25,900	0.18	Dec. 22	929,470	0.18
16	46,326	....	29	1,024,500	....
20	71,000	....	1861.		
24	101,760	....	Jan. 9	1,121,100	....
25	107,000	....	19	1,278,000	....
31	135,260	....	26	1,342,800	....
Sept. 1	140,500	....	Feb. 2	1,426,000	....
8	189,500	....	11	1,485,000	....
15	242,860	....	16	1,543,000	....
22	277,000	....	23	1,602,000	....
30	320,000	....	March 2	1,661,000	....
October 6	375,000	....	9	1,720,000	0.18
13	429,000	....	13	1,779,000	0.17
20	484,000	....	23	1,829,000	....
27	538,000	....	30	1,885,000	....
November 3	577,800	....	April 6	1,945,000	....
10	617,800	....	13	2,000,000	....
17	657,500	....	20	2,059,000	....
23	712,300	....	27	2,110,000	....
December 1	768,100	....	May 4	2,165,000	....
8	821,970	....	11	2,250,000	....
15	875,000	0.18	June	2,727,754	0.17

The piece had now received nearly 4,000,000 changes in all, but the 2,727,000 changes after it was once broken and repaired did not injure it. The changes were not very rapid. During the first experiment they averaged about 11,000 per day, or less than eight per minute, and during the last experiment the highest rate of change appears to have been less than eleven per minute, which is very slow compared with the strokes of some forge hammers.

189. STIFFENING UNDER STRAIN.—The experiments recorded in Articles 31, 185, and 186, indicate that iron may become stiffer, if not stronger, under strain. That such is the fact has recently been confirmed in a very striking manner by the experiments of Professor Thurston (hereafter given), and the following experiment, which was made by Commander L. W. Beardslee, of the U. S. Navy.*

* Reported by Prof. Thurston to the Am. Soc. of Civ. Eng., New York, Nov., 1874. The specimen is preserved in The Stevens' Institute of Technology.

The bar was of Phœnix iron, with an eye formed at each end. It was pulled apart by a hydraulic machine, and registered

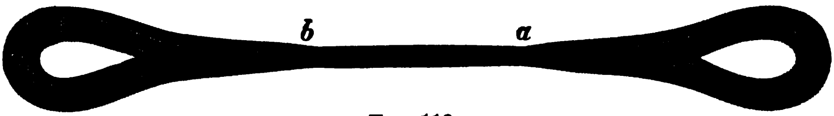

FIG. 116.

with weights and levers. It was originally $1\frac{3}{8}$ inches in diameter, but was drawn down to $1\frac{2}{8}$ of an inch at *a*, when it suddenly broke in the weld near the eye, with 67,800 pounds. The rest of the bar was slightly tapered, but with no marked diminution.

A new eye was welded on, and the next day it was put in the testing machine, when, instead of breaking at *a*, it began to yield and draw down at *b*, and finally broke at that point under a strain of 88,000 pounds.

Such are the *facts* in regard to this remarkable fracture. Had the eye not broken, it is quite certain that the bar would have broken at *a*. The particles then were moving (flowing) over each other more rapidly than at any other point, and were, apparently, on the verge of separation. By being relieved of the strain for a day (resting) its strength was greatly increased, so that it was stronger at the reduced section on the second day than the full section was on the former day. It would apparently have broken on the first day with a strain somewhat exceeding 68,000 pounds, but on the second day it sustained 88,000 pounds at that point without fracture, which is a very large increase in the strength.

The most that can be said with certainty is, that the particles by flowing over each other, and having time to come to rest in their new positions, the cohesive force between them was increased. The contraction necessarily develops heat, and it is probable that the heat during the flowing, and the abstraction of it afterwards, played an important part in securing the increased cohesion.

190. SHOCKS.—In a broad sense, a *shock* is the impinging of one mass against another, whereby the velocity of one or both of the masses is suddenly changed. In common language it is a *blow* produced by one *solid* body striking another. In the impact of gases, liquids and semi-fluids, *shocks* are not considered. When the motion of a rigid body is *gradually* changed, like the connecting rod, or the pitman of an engine, shocks are not produced. If a moving mass be brought to rest by a resistance acting through a finite space, the *shock* is much less than if it be arrested more suddenly. Thus a forge hammer in striking a molten mass of metal produces but little shock, whereas the same blow upon cold metal may produce a severe *shock*. No moving mass can be brought to rest *instantaneously*, but the more rigid the masses, and the more unyielding the supports of the body receiving the blow, the more suddenly will the blow be arrested, and the more severe will be the shock.

The effect of *shocks* may be greatly modified by the introduction of springs. Thus, the use of steel, rubber and wooden springs in vehicles and machines are familiar examples, and if the springs have but little mass, and have sufficient range of action, they may very nearly remove the effect which shocks would otherwise produce.

Oft-continued and long-repeated shocks upon metals are quite certain to produce fracture sooner or later. One who is unaccustomed to these effects is apt to be surprised at the failure of iron or steel after it has sustained a moderate shock for a long time, but those who are accustomed to them seek to anticipate and provide against them. All metals in use have their "life." In some cases they are worn out, but in many others they break after a time. They can sustain only a certain amount of service. All machinery, tools, implements, vehicles, etc., have to be renewed. But there is nothing more uniformly disastrous to machinery, or which produces results more unexpectedly than shocks.

The following example is a good illustration of its effects.

To aid in the handling of large masses of iron while being forged, a long bar of iron is sometimes forged to them to serve as a handle. This handle is called a "Porter bar," and may be used repeatedly for the same purpose.

At the West Point Foundry a Porter bar, which had been in use about twenty years, broke near the middle whilst the hammer was at work upon the forging which was attached to the other end. The bar was about twenty feet long and twelve inches in diameter at the smaller end, and twenty-three inches at the end where it was attached to the forging. It was about fourteen inches in diameter where it broke. It was slung on a chain in the usual manner, and the fracture was between the free end and where it was slung, and some two or three feet from the latter place.

The appearance of the fracture was described as highly crystalline and a clean break. The piece broken off probably weighed a ton and a half. It would have required a load of nearly fourteen tons applied at the end to have broken it if the iron was sound.

This is a remarkable fracture for iron. It is not probable that the iron was *crystallized*, but that it had that *appearance* on account of the character of the breakage, as will be explained hereafter.

The heavy end, which served as a handle, was caused to vibrate under the action of the hammer, and doubtless caused excessive strains which started a fracture; and by repeating the operation from time to time finally caused rupture.

The writer is familiar with similar examples in the case of steel. Where the steel had been subjected to repeated shocks, one end of a bar would drop off while the smith was at work upon the other end.

The fracture in such cases is doubtless a slow process. At first a mere crack is started, which increases slowly by the repeated blows, but is unseen by the observer until the piece is so much weakened that it fails suddenly at last.

The effect of a low temperature upon metals when subjected to shocks is not fully determined. When subjected to a steady tensile strain, numerous experiments prove conclusively that iron is stronger at very low temperatures than at ordinary temperatures. But it is commonly supposed that machinery, tools, rails on the railroad, tyres on locomotives, axles under the cars, etc., break more easily when cold than when warm. Steel rails when they first came into use were supposed to be more liable to break when cold than iron ones, but they have now come into extensive use, and there are no more breakages than formerly, and probably not as many.

Mr. Sandberg, the translator of Styffe's work,* thought it probable that iron when subjected to shocks might not give the same relative strength at different temperatures that it would when subjected to a steady strain. He therefore instituted a series of experiments to satisfy himself upon this important point, and aid in solving the problem. The following is an abstract of his report:—

The supports for the rails in the experiments were two large granite blocks which rested upon granite rocks in their native bed. The rails were supported near their ends on these blocks. They were broken by a ball which weighed 9 cwt., which was permitted to fall five feet the first blow, and the height increased one foot at each succeeding fall, and the deflection measured after each impact. A small piece of wrought-iron was placed on the top of each rail to receive the blow, so as to concentrate its effect.

The rail was thus broken into two halves, and each part was afterward broken at different temperatures. As the experiments were not made till the latter part of the winter, the lowest temperature secured was only 10° Fahr. Fourteen rails were tested:—Seven of which were from Wales; five from France; and two from Belgium. From these the experimenter drew the following conclusions:—

1. "That for such iron as is usually employed for rails in the three principal rail-making countries (Wales, France, and Belgium), the breaking strain, as *tested by sudden blows or shocks*, is considerably influenced by cold; such iron exhibiting at 10° F., only one-third to one-fourth of the strength which it possesses at 84° F.

2. "That the ductility and flexibility of such iron is also much affected by cold, rails broken at 10° F., showing on an average a permanent deflection of less than one inch, whilst the other halves of the same rails, broken at 84° F., showed less than four inches before facture."

These experiments seem to be conclusive for the iron which was tested.

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* The *Elasticity, Extensibility, and Tensile Strength of Iron and Steel.* By Knut Styffe. Translated by Christer P. Sandberg, London.

From the official reports of the *Verein Deutscher Eisenbahn Verwaltungen*, it appears that during the year 1870, on 22 lines belonging to the Association, 132 axles of locomotives, tenders, and carriages were broken. In comparison with the previous year, in which the fractures amounted to 163, these figures show an improvement. There is a decrease of 19.3 per cent. on the service which, considering the extraordinary demands occasioned by the Franco-German war, and the increase of rolling stock in Austria, appears considerable. The fractures either occurred or were reported in the months of

December, January, February,		in 39	cases.
March to May	(inclusive),	in 30	"
June to August,	"	in 25	"
September to November	"	in 38	"

The influence of the cold season, despite much that has recently been said to the contrary, is distinctly marked; from March till August 55 only, and during the other months 77 axles broke. The average run of the axles broken in 1870 was as follows:—

Locomotives 11	years 4	months	13 days.
Tenders.............13	" 4	"	20 "
Carriages............11	" 11	"	13 "
Average.............12	" 2	"	29 "

The average mileages of the axles were in the case of

Locomotives 34,241.7	miles	(German).
Tenders....31,494.5	"	"
Carriages...24,040.1	"	"
Average....27,631.1	"	"

The maximum mileage attained was 69,000 miles.

But in opposition to this we have the Report of the Massachusetts Railroad Commissioners for 1874. On page 74 of this report are the following conclusions:—" Cold does not make iron or steel brittle, or unreliable for mechanical purposes." " It is not the rule that the most breakages occur on the coldest days." " The introduction of steel, in place of iron rails, has caused an almost complete cessation of the breakage of rails."

This report, which is the latest upon this point, shows that there must have been a great improvement in some respects in order to secure it.

These results are in opposition to previously formed ideas in regard to the effect of cold. It being the latest report, and from a reliable source, we must look for an explanation in the improved character of the materials in the rails or in the substructure. There doubtless remains much to be learned upon this subject. It is especially desirable to determine the effect of the *impurities* in the metal. It is probable that those elements which make iron cold-sheet will cause it to be more brittle at low than at moderate temperatures; and that *good* metals will resist shocks better at low temperatures than at moderate ones.

The following experiments, by John A. Roebling,* bear upon this subject:

"The samples tested were about one foot long, and were reduced at the centre to exactly three-fourths of an inch square, and their ends left larger, were welded to heavy eyes, making in all a bar three feet long. These were covered with snow and ice, and left exposed several days and nights. Early in the morning, before the air grew warmer, a sample inclosed in ice was put into the testing-machine and at once subjected to a strain of 26,000 pounds, the bar being in a vertical position, and left free all around. The iron was capable of resisting 70,000 lbs. to 80,000 lbs. per square inch. A stout mill-hand struck the reduced section of the piece, horizontally, as hard as he could, with a billet one and a half inches in diameter and two feet long. The samples resisted from three to one hundred and twenty blows. With a tension of 20,000 lbs. some good samples resisted 300 blows before breaking."

The finest and best qualities of iron, or those that have the highest coefficient elasticity will resist vibration best. It is generally supposed that good iron will resist concussions much better than steel. Sir William Armstrong, of England, says:—
"The conclusion at which I have long since arrived, and which I still maintain, is, that although steel has much greater tensile

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* *Jour. Frank. Inst.*, vol. xl., 3d series, p. 361.

strength than wrought iron, it is not as well adapted to resist concussive strains." This was written many years since, but at the present day many mechanics prefer iron to steel for resisting shocks.

191. CRYSTALLIZATION OF IRON.—It is observed that metals which are subjected to oft-repeated and long-continued shocks become weak; and when broken in this way they appear to be crystallized, having *apparently* undergone a change of structure. A crystal is a homogeneous inorganic solid, bounded by plane surfaces, systematically arranged. The quartz crystal is a familiar example. Different substances crystallize in forms which are peculiar to themselves. Metals, under certain circumstances, crystallize; and if they are broken when in this condition the fracture shows small plane surfaces, which are the faces of the crystals. It is found in all cases that *crystallized iron is weaker than the same metal in its ordinary state.* By its ordinary state we mean that wrought iron is fibrous, and cast iron and steel are granular in their appearance.

Iron crystallizes in the cubical system.* Whöler, in breaking cast-iron plates readily obtained cubes when the iron had long been exposed to a white heat in the brickwork of an iron smelting furnace.

Augustine found cubes in the fractured surface of gun barrels which had long been in use.

Percy found on the surface and interior of a bar of iron, which had been exposed for a considerable time in a pot of glass-making furnace, large skeleton octahedra. (He seems to differ from the preceding in regard to the form of crystals.)

Prof. Miller, of Cambridge, found Bessemer iron to consist of an aggregation of cubes.

Mallet says:—"The plans of crystallization group themselves perpendicular to the external surfaces."

Bar iron will become crystalline if it is exposed for a long time to a heat considerably below fusion. Hence we see why large masses which are to be forged may become crystalline,

* Osborn's *Metallurgy*, pp. 83–86. See Appendix.

on account of the long time it takes to heat the mass. Forging does not destroy the crystals, and forging iron at too low a temperature makes it tender, while steel at too high a temperature is brittle. The presence of phosphorus facilitates crystallization.

Time, in the process of breaking iron, will often determine the character of the fracture. If the fracture is slow, the iron will generally appear fibrous; but if it be quick, it may appear more or less crystalline. This result has been frequently noticed. At Shoeburyness armor-plates were shattered like glass under the impact of shot at a velocity of 1,200 feet to 1,600 feet per second. They were were made of good fibrous iron.

William Fairbairn says:—* "We know that in some cases wrought iron subjected to continuous vibration assumes a crystalline structure, and that then the cohesive powers are much deteriorated; but we are ignorant of the causes of this change."

The late Robert Stephenson † stated that in all the cases investigated by him of supposed change of texture, he knew of no single instance where the reasoning was not defective in some important link.

Mr. Brunel accepted the theory of molecular change, for a time, as due to shocks, but afterwards expressed great doubts as to its correctness, and thought that the appearance depended more upon the manner of breaking the metal than upon any molecular change.

Fairbairn presented his view of the probable cause of the internal change when it takes place in *his evidence before the Commissioners appointed to inquire into the application of iron to railway structure.* He says:—"As regards iron it is evident that the application and abstraction of heat operates more powerfully in effecting these changes than probably any other agency; and I am inclined to think that we attribute too much influence to percussion and vibration, and neglect more obvious causes which are frequently in operation to produce the change. For example, if we take a bar of iron and heat it red hot, and then plunge it into water, it is at once converted

* *Civ. Eng. and Arch. Jour.*, Vol. iii., p. 257.

† *Am. R. Times*, March 6, 1869, Boston.

into a crystallized instead of a fibrous body; and by repeating this process a few times, any description of malleable iron may be changed from a fibrous to a crystalline structure. Vibration, when produced by the blows of a hammer or similar causes, such as the percussive action upon railway axles, I am willing to admit is considerable; but I am not prepared to accede to the almost universal opinion that granulation is produced by those causes only. I am inclined to think that the injury done to the body is produced by the weight of the blow, and not by the vibration caused by it. If we beat a bar with a small hammer, little or no effect is produced; but the blows of a heavy one, which will shake the piece to the centre, will probably give the key to the cause which renders it *brittle*, but probably not that which causes *crystallization.* The fact is, in my opinion, we cannot change a body composed of a fibrous texture to that of a crystalline character by a mechanical process, except only in those cases where percussion is carried to the extent of producing considerable increase of temperature. We may, however, shorten the fibres by continual bending, and thus render the parts brittle, but certainly not change the parts which were originally fibrous into crystals.

"For example, take the axle of a car or locomotive engine, which, when heavily loaded and moving with a high velocity, is severely shocked at every slight inequality of the rails. If, under these circumstances, the axle bends—however slightly—it is evident that if this bending be continued through many thousand changes, time only will determine when it will break. Could we, however, suppose the axle so infinitely rigid as to resist the effects of percussion, it would then follow that the internal structure of the iron will not be injured, nor could the assumed process of crystallization take place."

The late John A. Roebling, who designed and constructed the Niagara Railway Suspension Bridge, in his report on that structure in 1860,* says he has given attention to this subject for years, and as the result of his observation, study and experiment, gives as his view that "a molecular change, or so-called *granulation* or *crystallization*, in consequence of vibration or

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* *Jour. Frank. Inst.*, Vol. xl., 3d series, p. 361.

tension, or both combined, has in no instance been satisfactorily proved or demonstrated by experiment." "I further insist that crystallization in iron or any other metal *can never take place in a cold state.* To form crystals at all, the metal must be in a highly-heated or nearly molten state." But he states that he is witnessing the fact daily that vibration and tension combined will greatly affect the strength of iron *without changing its fibrous texture.*

In speaking of the rock-drilling engines used in Hoosac Tunnel, Mass., which were driven by compressed air, the committee says: *—"Gradually they began to fail in strength; the incessant and rapid blows—counted by millions—to which they are subjected, appearing to *granulate* or *disintegrate* portions of the metals composing them."

In some recent experiments made in France, interesting information has been made known in regard to crystalline structure in wrought-iron. The apparatus consisted of a bent axle, which was firmly fixed up at the elbow in timber, and which was subjected to torsion or twist by means of a cog-wheel connected at the end of the horizontal part. At each turn the angle of torsion was 24 degrees, and a shock was produced each time that the bar left one tooth to be raised by the next. Seven axles were submitted to the trial. In the first the movement lasted one hour, 10,800 revolutions and 32,400 shocks being produced; the axle, $2\frac{6}{10}$ inch diameter, was taken from the machine and broken by an hydraulic press, but no change in its texture had occurred. In the second, a new axle having been tried 4 hours, sustained 129,000 torsions, and was afterwards broken by means of an hydraulic press; no alteration was perceptible to the naked eye, but, tried by a microscope, the fibres appeared without adhesion, like a bundle of needles. A third axle was subjected during 12 hours to 338,000 torsions, and broken in two; a change in its texture and an increased size in the grain of the iron were observed by the naked eye. In the fourth, also, the axle was broken in many places after 110 hours and 2,553,000 torsions. In the fifth, an axle submitted to 23,328,000 torsions during 720 hours, was completely changed in its texture.

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* *Annual Report of the Commissioners on the Troy and Greenfield Railroad and Hoosac Tunnel.* House Doc., No. 30, p. 5, Boston, Mass.

In the sixth, after ten months, during which the axle was submitted to 78,735,000 torsions and shocks, fracture produced by an hydraulic press showed clearly an absolute transformation, the surface of the rupture being scaly, like pewter. The seventh axle, submitted to 129,304,900 torsions, presented a surface of rupture like that of the sixth, the crystals were found to be perfectly well-defined, it having lost every appearance of wrought iron.—*U. S. R. R. and Mining Register*, 1872.

The last experiment looks like a *proof* of the fact that the internal *structure* of iron may be changed by strains and shocks, but in this example millions of them did not produce rupture. Much depends upon the severity of the shock. The rapidity of the blows also has much to do with its durability, since a rapid movement of the particles may develop heat to such an extent as to become an important element in the effect produced upon the metal.

These several facts, though apparently somewhat conflicting, show quite conclusively, that some metals will crystallize under certain conditions; that under certain conditions they may be strained millions of times without being damaged, or at least without being broken; that under certain conditions strains and shocks combined may produce crystallization; that shocks when severe will weaken metals, and if they are sufficiently numerous, will produce rupture. Much evidently remains to be learned upon this subject. There is a metal called "Phospho-Bronze," which combines in a remarkable degree toughness, rigidity, hardness and great elastic resistance, which, it is said, will not crystallize under repeated strains or continued vibration.

192. THE PRACTICAL QUESTION is, how shall the life of such machines as are necessarily subjected to shocks, be prolonged. The steam forge hammer (Nasmyth's) has been very troublesome on account of its frequent breakages. The automatic valve arrangement was so troublesome that many preferred to work them entirely by hand, but at the present day there are many in which this is as durable as any other part of the machine. One of the essential features is to take the motion off from the hammer or piston-rod by a slope, so that the movement of the valve and its mechanism will be gradual. The piston-rod is liable to break. In some very heavy hammers the rod is keyed to the block and the end, which is square, presses against blocks of wood which are put in place for the purpose of relieving the shock, but this only partially cures the evil, since the rod is liable to break at the keyhole.

A Mr. Webb, of England, proposed the improvement shown in Fig. 117.

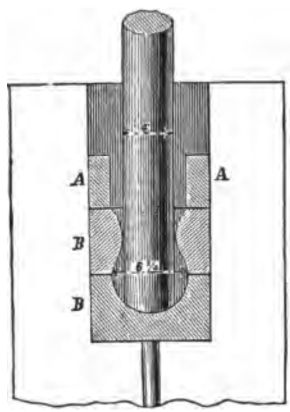

FIG. 117.

Referring to the figure, it will be seen that the piston-rod, which is for the main part of its length 4 in. in diameter, is enlarged at the lower end to 6¾ in. in diameter, and is shaped spherically. This spherical portion of the rod is embraced by the annealed steel castings, B B, which are secured in their place in the hammer-head by the cotters, A, and the whole thus forms a kind of ball-and-socket joint, which permits the hammer head to swivel slightly on the rod without straining the latter. Mr. Webb first applied this form of hammer-rod fastening to a five-ton Nasmyth hammer with a 4 in. rod. With the old mode of attachment, with a cheese end, this hammer broke a rod every three or four weeks when working steel, while a rod with the ball-and-socket joint, which was put in in November, 1867, has been working ever since, that is, to some time in 1869, without giving any trouble. The inventor has also applied a rod thus fitted to five-ton Thwaites and Carbutt's hammer with equal success.

One Morrison avoided the difficulty by making the rod *very* large and of uniform size, from the piston down to the hammer face.

Mr. Samuel Tretheweỵ, of Pittsburgh, Pa., thought, by reducing the rod at A, that he would compel the breakage to take place at that point, and that the repairs could be more quickly made than when the breakage was permitted to take place at any point; but, to his surprise and gratification, the rods lasted from two to three times as long without breaking as they did when the rods were of uniform size. The ends taper one-half inch to twelve inches.

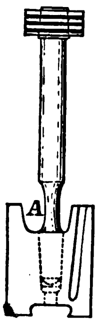

FIG. 118.

Steam rock-drilling machines are of more recent date than steam hammers, but partake of the same difficulties, and many more in addition to them. They must be portable, and hence comparatively light; but they have severe work to do, and hence should be very strong. But for durability they *must* have *mass*.

One of the ways of making such machines successful is to learn by practical experience where they are liable to break, and provide them with duplicate parts.

Another efficient way of improving them is to make some simple non-expensive part, such as a bolt, pin, rod or bar, comparatively weak, so that it will break first. The main parts will thus be preserved, and an ordinary mechanic may make the repairs. The use of a wooden pin for connecting the parts which would break when the machine met with a serious obstruction, has greatly prolonged the life of certain machines. All reënterent sharp angles should be avoided in machines subjected to shocks.

The cause of breakages has sometimes been attributed to crystallization, when the true cause was a lack of strength. In case of percussive forces the strain may exceed the amount estimated, and thus damage the material. As a general rule in such cases, the greater the amount of metal in the structure, when properly proportioned, the longer will be its life.

The *life of metals*, or the amount which they will endure in performing a certain duty, is being determined approximately by actual use. Having determined it, other specimens of similar quality, when used for the same purpose, should be cast aside before they fail, after having performed nearly the same duty.

193. THURSTON'S EXPERIMENTS.*—Professor Thurston, of the Stevens' Institute of Technology, has made an entensive series of experiments upon various materials with a machine of his own invention, the prominent feature of which is its automatic registry. For the sake of simplicity, compactness, and economy, he so constructed the machine as to subject the

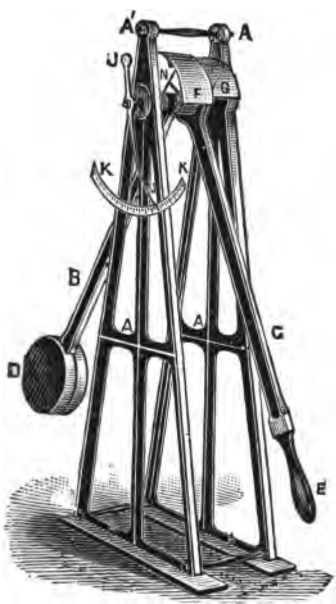

FIG. 119.

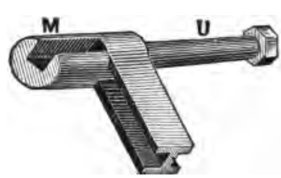

FIG. 120.

specimens to torsion. It records automatically at every instant the moment of the stress, and the total angle of torsion. This feature enables one to make experiments rapidly and accu-

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* Papers read before the Am. Civ. Eng. Society, N. Y., *Jour. of the Frank. Inst.* 1874.

rately, and by means of it many qualities may be detected which otherwise might escape observation.

The twisting force is applied at *E*, Fig. 119, and the resistance is offered by a weight *D*. The arms *B* and *C* turn upon independent axes, an enlarged view of one of which is shown in Fig. 120. The end *U* swings in the frame, while the other end is free. At the free end is a rectangular recess *M* for receiving one end of the specimen, which is usually made of the form shown in Fig. 121. The reduced part is one inch long and five-eighths of an inch in diameter. The other axis faces this, but a short space is left between the free ends at *M*.

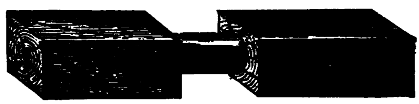

FIG. 121.

When the specimen is secured in the rectangular recesses, the axes are virtually connected by the specimen, so that as a force is applied to the arm *E*, tending to turn it on its axis, it will at the same time tend to turn the arm *B* on its axis; but as the weight *D* is moved from the vertical position it will bring a torsive strain on the specimen, and the farther it is forced out the greater will be the strain. The statical moment of the weight *D* will equal the moment of the torsional stress. The relative angular movements of the arms *C* and *B* will be the measure of the total angle of torsion. It is evident that as the specimen yields to the strain, the arm *C* must travel farther than the arm *B*, in producing a given strain.

A guide curve *F*, of such form that its ordinates are proportional to the torsional moments, and its abscissas proportional to the arcs moved over by the arm *B*, is attached to the frame *AA'*. The other arm *C* carries a cylinder *G*, upon which paper is clamped for receiving the record. A pencil is secured to the arm *B* in such a way that it will be carried around with it, but which, at the same time, is free to move outward as it is moved along the curve *F*.

After the specimen paper and pencil are arranged, the arm *C* is forced around, and arm *B* is thus forced forward, and the pencil describes a line upon the paper as they move. The abscissa of the line will represent the angle of torsion and the ordinates the moment of stress.

The interpretation of the diagrams has been the subject of much study. The comparison of the diagrams of a variety of materials, and a knowledge of the properties of some of them, enables the investigator to draw many general conclusions. Those who desire to acquaint themselves with the steps should consult the original papers. We can only present the results, and for this purpose have selected the diagram of a specimen of Swedish iron, marked No. 101, which is an exact copy of the diagram made by the machine. After becoming familiar with this one, the student will be able to interpret any of those upon the accompanying chart.

1. The total angle of torsion is marked on the lower horizontal line.

2. The ordinates, as bx, represent the moments of torsion, and hence represent relatively the strain.

3. The curve from A to b is convex towards the line of abscissas. This shows that the piece had internal strains before it was twisted.

4. From a to b the line is very nearly straight, and the tangent of its inclination to the horizontal is the ratio of the moment of torsion to the total angle of torsion (or distortion) which took place between a and b. When the line is exactly straight, Hooke's law, "*ut tensio sic vis*," is mathematically exact. The inclination of the line is a measure of the stiffness, and is proportional to the coefficient of elasticity. The point b, where the line begins to curve, corresponds to the limit of elasticity. When there is no straight part, as in many specimens of cast-iron and some other metals, there is not properly any limit of elasticity.

5. The curve, at b, shows that the outer particles begin to yield, and a set takes place.

6. The relative depression at c shows that the *structure* of the material is not homogeneous. Homogeneous materials make a nicely-rounded curve without any depression. This simple illustration shows one of the advantages of an automatic registry.

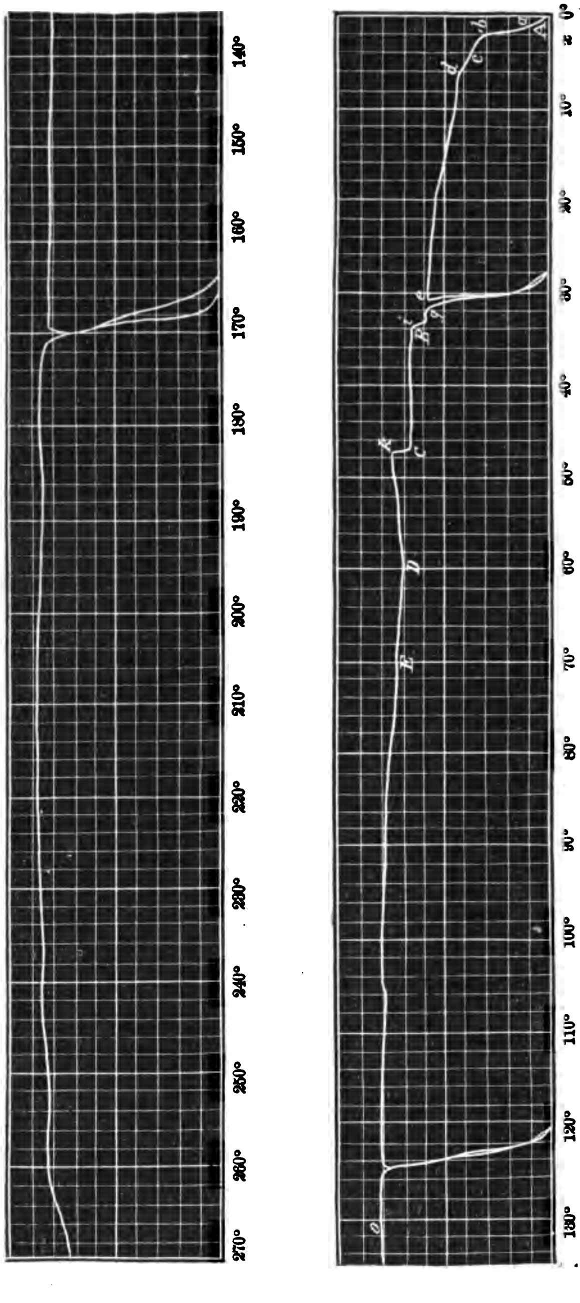

Diagram of strains of a piece of Swedish iron, marked 101 on the accompanying Plate, as determined by subjecting the piece to torsion. The moments of strains, in foot-pounds, are represented by the ordinates to the curve, and the angle of torsion by the abscissas.

FIG. 122.

7. The curve rises more rapidly from *c* to *d*, showing that the previous strains upon the non-homogeneous solid has finally brought into action, or at least into greater action, certain elements which before were partially dormant.

8. From *d* to *e* the curve is very regular, showing that the yielding goes on uniformly.

9. At *e* the motion of the lever *C* was reversed until the arm *B* returned to a vertical position, and the pencil traced the curve *ef*. A forward movement was then given to the arms, and the pencil traced the curve *fg*. The abscissa *Af* represents the set. The inclination of the curve *gf*, as before, represents the stiffness at this point. In this case the stiffness is not diminished by the previous strains, and the elastic limit has been raised.

10. The strain being continued, the curve *gB* maintains the same height as before, showing that the strength has not been impaired; but at *B* there is a slight depression. At *B* the arms were fixed in position and the piece was left under strain for one day, when the arm *C* was again moved forward, and the line moved suddenly upward to *i*.

11. The elevation of the curve *Bi* was an unexpected discovery by the experimenter, which was first formally announced by him to the American Society of Civil Engineers in November, 1873. The explanation of this phenomenon is given in his own words, as follows: "*The phenomenon here observed is an elevation of the limit of elasticity by a continued strain. The cause is probably a gradual release of internal strain*, occurring in a somewhat similar manner to that observed previously in cast-iron, and less frequently, and generally in a less marked degree, in wrought-iron and other metals, which have been worked in large pieces, and in which such strain has been more or less reduced by a period of rest."

The piece resisted a much larger strain within the elastic limits, after resting twenty-four hours, than it did before; and it evidently required a greater force to rupture it than it would if the experiment had continued consecutively on the first day. The experimenter, however, concludes that it would probably rupture with a less angle of torsion after resting than it would have done without resting. We have here a clear proof of the vague inferences which might be drawn from Articles 35, 187, and 188. The experiments of Commander Beadsley, which were made at a later date, by direct tension, are a striking confirmation of this phenomenon, as given in Article 189.

12. The curve *iC* being nearly horizontal shows that the resistance remains nearly constant whilst the twisting force is active, and the torsional angle is constantly increasing. This corresponds to the gradual pulling apart of pliable bodies under the action of a nearly constant force, the elongation gradually increasing while the section is gradually diminishing, until it finally breaks.

13. At *C* the strain was maintained constant for another day, when we see that another elevation in the elastic limit from *C* to *k* took place.

14. From *k* the resistance gradually diminished to *D*, after which it gradually increased to *E*. At this point a sudden movement was given to the lever *C*, and immediately after the movement ceased there was a slight depression in the curve, after which the force was gradually applied and the height of the curve was gradually restored. The depression in this diagram is small compared with those which arose from a similar cause in No. 118, as shown on the plate at the points *be*, *b′c′*, and *gh*.

This shows that materials will not resist as much to a sudden force as they will to one which is applied more gradually. This at first appears paradoxical, since it seems to require a larger load to break it in a longer time. But in the light of the preceding experiments, and those before referred to, it can be easily explained. Suppose, for instance, that 20,000 pounds is just sufficient to break a piece if applied all at once; then if 18,000 be applied, and after a few hours 500 pounds more be added, and after another interval 500 pounds more are added, and so on, it will be found that the piece will sustain much more than 20,000 pounds under the process of slow loading.

It thus appears that the load which is applied when the action is rapid, is not a true measure of the strain; the more rapid the action the less being the strain.*

* This may be aptly illustrated by a dynamical problem. If a chord passes over a pulley, and 100 pounds is attached to each end of it, the tension on the

This principle has an important bearing upon the effect of shocks upon machinery. It shows that they are not only weakened by shocks, as stated in previous articles, but that the material is inherently weaker. It shows that the *resilience*, or its resistance to shock, cannot be correctly determined from the work of a statical load in producing rupture.

After the strain passes the point E, it is not perceptibly diminished until it has been twisted through an angle of 220 degrees, after which it gradually fails and finally breaks.

15. The parallelism of the elastic lines shows that the elasticity remains quite unimpaired up to the point of incipient rupture, a fact which was observed by some of the earliest experimenters in this field of investigation.

The principles here stated will enable the student to determine the qualities of the several specimens shown on the accompanying Plate.

16. The fact that the resistance remains so nearly uniform while the torsional angle is increasing so largely, leads one to infer that as the outer elements become weakened by being overstrained, that those near the axis resist more, and there appears to be a tendency to cause all the elements to resist the same amount; so that at the instant of rupture, in ductile bodies, the greater part of the transverse section resists uniformly. It is evident that the law given in the Chapter on Torsion, that the resistance varies directly as the distance from the axis, is not true after a set has taken place. If the resistance is uniform, we have, in the case of cylinders, $\pi r^2.J$ for the total resistance, in which J is the modulus for ultimate shearing, which, in the case of wrought-iron, is nearly the same as T, the tenacity. The mean arm of this force is $\frac{2}{3}r$; hence, we have

$$Pa = \tfrac{2}{3}\pi r^3 J.$$

Equation (202) is $Pa = \frac{1}{2}\pi r^3 J$.

The former is $1\frac{1}{3}$ times the latter.

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chord will be 100 pounds. But if 50 pounds be removed from one end, accelerated motion will at once take place, and the tension will no longer be 100 pounds, but it will be $66\frac{2}{3}$ pounds. This is considerably less than the 100 pounds, the greater weight. The greater the acceleration the less will be the tension compared with the load.

But as it is not probable that the strains can be made uniform, the former may be considered the superior limit of the strength.

17. The diagrams on the upper half of the Plate represent the strains on several specimens at various temperatures. The general conclusion arrived at by these experiments was, that with pure, well-worked metals, a diminution of temperature produces an increase of strength, but when there is an excess of impurities this law may be reversed, especially in case of shocks.

CHAPTER XI.

LIMITS OF SAFE LOADING OF MECHANICAL STRUCTURES.

194. RISK AND SAFETY.—We have now considered the breaking-strength of materials under a variety of conditions, and also the changes produced upon them when the strains are within the elastic limits. In a mechanical structure, in which a single piece, or a combination of pieces, are required to sustain a load, it is desirable to know how small the piece, or the several pieces, may be made to sustain a given load *safely* for an indefinite time; or, how much a given combination will sustain *safely*. The nature of the problem is such that an exact limit cannot be fixed. Materials which closely resemble each other do not possess *exactly* the same strength or stiffness; and the conditions of the loading as to the amount or manner in which it is to be applied, may not be exactly complied with. Exactness, then, is not to be sought; but it is necessary to find a limit below which, in reference to the structure, or above which, in reference to the load, it is not safe to pass.

It is evident that to secure an economical use of the material on the one hand, and ample security against failure on the other, the limit should be as definitely determined as the nature of the problem will admit; but in any case we should incline to the side of safety. No doubt should be left as to the stability of the structure. *There is no economy in risk* in permanent structures. Risk should be taken only in temporary, or experimental, structures; or where risk cannot, from the nature of the case, be avoided.

195. ABSOLUTE MODULUS OF SAFETY.—In former times, one of the principal elements which was used for securing safety in a structure, was to assume some arbitrary value for the resistance of the material, such value being so small that the material could, in the opinion of the engineer, safely sustain it. This is a convenient mode, but very unphilosophical, although still extensively used. The plan was to determine, as nearly as possible, what good materials would sustain for a long period, and use that value for all similar materials. But it is evident, from what has been said in the preceding pages, that some materials will sustain a much larger load than the average, while others will not sustain nearly so much as the average. In all such cases the proper value of the *modulus* can only be determined by direct experiment. In all important structures the strength of the material, especially iron and steel, should be determined by direct experiment.

The following values are generally assumed for the *modulus of safety*.

		Pounds per square inch.
Wrought-iron, *for tension or compression*, from		10,000 to 12,000
Cast-iron, for *tension*, from		3,000 to 4,000
Cast-iron, for *compression*, from		15,000 to 20,000
Wood, *tension or compression*, from		850 to 1,200
Stone, *compression*	granite, from	400 to 1,200
	quartz, from	1,200 to 2,000
	sandstone, from	300 to 600
	limestone, from	800 to 1,200

The practice of French engineers,* in the construction of bridges, is to allow 3.8 tons (gross) per inch upon the gross section, both for tension and compression of wrought iron.

The Commissioners on Railroad Structures, England, established the rule that the maximum tensile strain upon any part of a wrought iron bridge should not exceed five tons (gross) per square inch.†

In most cases the *effective section* is the section which is subjected to the strain considered.

196. FACTOR OF SAFETY.—The next mode, and one which is also largely in use, is to take a fractional part of the ultimate strength of the material for the limit of safety. The reciprocal of this fraction is called *the factor of safety*. It is the ratio of the ultimate strength to the computed strain, and hence is

* *Am. R. R. Times*, 1871, p. 6.

† *Civ. Eng. and Arch. Jour.*, Vol. xxiv., p. 327.

the *factor* by which the computed strain must be multiplied to equal the actual strength of the material, or of the structure.

Experiments and theory combine to teach that the *factor of safety* should not be taken as small as 2. See articles 19, 185, 186, and 188.

Beyond this the *factor* is somewhat arbitrarily assumed, depending upon the ideas of the engineer. For instance, the following values were given to the Commissioners on Railway Structures in England.*

	Factors.
Messrs. May and Grissel	3
Mr. Brunel	3 to 5
Messrs. Rasbrick, Barlow and others	6
Mr. Hawkshaw	7
Mr. Glyn	10

The following values are also given by others:—

		Factors.
Bow, for wrought-iron beams		3.5
Weisbach, for wrought-iron †		3 to 4
Vicat, for wire suspension bridges		more than 4
Rankine, for wire bridges	steady strain	3 to 4
	moving load	6 to 8
Fink, iron-truss bridges	for posts and braces	5 to 6
	for cast-iron chords	7
Fairbairn, for cast-iron beams ‡		5 to 6
C. Shaler Smith, compression of cast-iron		5
Rankine and others, for cast-iron beams		4 to 6
Mr. Clarke, in Quincy Bridge, lower chord		6 to 7
Washington A. Roebling, for suspension cables		6
Morin, Vicat, Weisbach, Rondelet, Navier, Barlow, and many others, say that for a wooden frame it should not be less than		10
For stone, for compression		10 to 15

From the experiments which are recorded in Article 188, Fairbairn deduced the following conclusions in regard to beams

* *Civ. Eng. and Arch. Jour.*, Vol. xxiv., p. 327.

† *Weisbach, Mech. and Eng.*, Vol. i., p. 201.

‡ *Fairbairn, Cast and Wrought-Iron*, p. 58.

and girders, whether plain or tubular.* "The weight of the girder and its platform should not in any case exceed one-fourth the breaking weight, and that only one-sixth of the remaining three-fourths of the strength should be used by the moving load." According to this statement the maximum load, including the live and dead load, may equal, but should not exceed,

$$\tfrac{1}{4} + \tfrac{1}{6} \text{ of } \tfrac{3}{4} = \tfrac{3}{8}$$

of the breaking load. Hence the *factor of safety* must not be less than 2.66 when the above conditions are fulfilled. This value is, however, evidently smaller than is thought advisable by most engineers.

The rule adopted by the Board of Trade, England, for railroad bridges is † "to estimate the strain produced by the greatest weight which can possibly come upon a bridge throughout every part of the structure which should not exceed *one-fifth the ultimate strength of the metal.*" They also observed that ordinary road bridges should be proportionately stronger than ordinary railroad bridges.

197. RATIONAL LIMIT OF SAFETY.—It is evident that materials may be strained any amount within the elastic limit. Their recuperative power—if such a term may properly be used in connection with materials—lies in their elasticity. If that is damaged the life of the material is damaged, and its powers of resistance are weakened. As we have seen in the preceding pages, there is no known relation between the coefficient of elasticity and the ultimate strength of materials. The coefficient of elasticity may be high and the modulus of strength comparatively low. In other words, the limit of elasticity of some metals may be passed by a strain of less than one-third their ultimate strength, while in others it may exceed one-half their ultimate strength. We see, then, the unphilosophical mode of fixing an *arbitrary modulus of safety*, or even a *factor of safety*, when they are made in reference to the ultimate strength. But an examination of the results of experiments

* *Civ. Eng. and Arch. Jour.*, Vol. xxiv., p. 329.
† *Civ. Eng. and Arch. Jour.*, Vol. xxiv., p. 226.

shows that the limit of elasticity is rarely passed for strains which are less than one-third of the ultimate strength of the metal, and hence, according to the views of the engineers given in the preceding article, the *factors of safety*, which are commonly used in practice, are generally safe. But if the limit of elasticity were definitely known it is quite possible that a smaller *factor of safety* might sometimes be used.*

This method of determining the limit has been recognized by some writers, and the propriety of it has been admitted by many practical men, but the difficulty of determining the elastic limit has generally precluded its use. The experiments which are necessary for determining it are necessarily more delicate than those for determining the ultimate strength.

In regard to the margin that should be left for safety, much depends upon the character of the loading. If the load is simply a dead weight, the margin may be comparatively small; but if the structure is to be subjected to percussive forces or shocks, it is evident, as indicated in articles 19 and 193, that the margin should be comparatively large, on account of the indeterminate effect produced by the force. In the case of railroad bridges, for instance, the vertical posts or ties, as the case may be, are generally subjected to more sudden strains due to a passing load, than the upper and lower chords, and hence should be relatively stronger. The same remark applies to the inclined ties and braces which form the trussing; and to any parts which are subjected to severe local strains.

In machines which are subjected to a constant jar while in use, it is very difficult to determine the proper margin which is consistent with economy and safety. Indeed, in such cases, economy as well as safety generally consists in making them *excessively* strong, as a single breakage may cost much more than the extra material necessary to fully insure safety.

The mechanical execution of a structure should be taken into consideration in determining the proper value of the margin of safety. If the joints are imperfectly made, excessive

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* James B. Eads, in his *Report upon the Illinois and St. Louis Bridge*, for 1871, states that he tested samples of steel which were to be used in that structure, which showed limits of elastic reaction of 70,000 to 93,000 pounds per square inch.

strains may fall upon certain points, and to insure safety the margin should be larger. No workmanship is *perfect*, but the elasticity of materials is favorable to such imperfections as *necessarily* exist; for, when only a portion of the surface which is intended to resist a strain is brought into action, that portion is extended or compressed, as the case may be, and thus brings into action a still larger surface.

198. EXAMPLES OF STRAINS THAT HAVE BEEN USED IN PRACTICAL CASES.—The margin of safety that has been used in various structures may or may not serve as guides in designing new structures. If the margin for safety is so small that the structure appears to be insecure and gives indications of failure, it evidently should not be followed. It serves as a warning rather than as a guide. If the margin is evidently excessively large, demanding several times the amount of material that is necessary for stability, it is not a guide. Any engineer or mechanic, without regard to scientific skill or economy in the use of materials, may err in this direction to any extent. But if the margin appears reasonably safe, and the structure has remained stable for a long time, it serves as a valuable guide, and one which may safely be followed under similar circumstances. Structures of this kind are practical cases of the approximate values of the inferior limits of the *factors of safety*. The following are some practical examples:—

IRON TRUSSED BRIDGES.

NAME OF THE BRIDGE.	TENSION. Tons per square inch.	COMPRESSION. Tons per square inch.
Passaic (*Lattice*)	$5\frac{1}{2}$ to 6	$4\frac{1}{4}$ to $5\frac{1}{4}$
Place de l'Europe (*Lattice*)	4	$3\frac{5}{8}$
Canastota (*N. Y. C. R. R.*) (*Lattice*)	5	4
Newark Dyke (*Warren Girder*)	5	5
Boyne Viaduct (*Lattice*)	5	
Charing Cross (*Lattice*)	5	4
	Pounds per square inch.	Pounds per square inch.
St. Charles, Mo. (*Whipple Truss*)*	12,000	12,000
Louisville, Ky. (*Fink Truss*)	7,000 to 12,000	$\frac{1}{6}$ to $\frac{1}{7}$ the strength
Keokuk and Hannibal †	9,251	8,962
Quincy Bridge ‡	10,000	Factor of safety, 5
Kansas City Bridge §	11,375	711
Hannibal Bridge (*Quadrangular Truss*) ‖	Factor of safety, 5	Factor of safety, 5

WOODEN BRIDGES.

NAME OF THE BRIDGE.	MAXIMUM STRAINS.
Cumberland Valley R. R. Bridge	635 pounds per square inch.
Portage Bridge (*N. Y. & E. R. R.*)	Factor of safety, 20.

* *R. R. Gazette*, July 8, 1871, p. 169.

† *R. R. Gazette*, July 15, 1871, p. 178. Pivot span 376 feet 5 inches: longest pivot span yet constructed.

‡ *Report of Chief Engineer Clark.*

§ Calculated from the *Report of Chief Engineer O. Chanute*, pp. 106 and 136.

‖ The tensile strength of the material ranged from 55,000 lbs. to 65,000 lbs. per square inch.—*R. R. Gazette*, July 15, 1871, p. 169.

CAST-IRON ARCHES.*

NAME OF THE ARCH.	SPAN.		VERSED SINE.		STRAIN PER SQUARE INCH IN TONS.
	Feet.	Inches.	Feet.	Inches.	
Austerlitz	186	0	10	7	2.78
Carrousal	152	2	16	1	1.46
St. Denis	102	5	11	4	1.37
Nevers	137	9	15	0	1.90
Rhone	197	10	16	5	2.37
Westminster	120	0	20	0	3.00

STONE ARCHES.†

NAME OF THE ARCH.	Span in feet.	Versed sine in feet.	Pressure per square inch in pounds at the key.	Factor of safety at the point of greatest strain.
Wellington	100	15	175	11.3
Waterloo (9 *Arches*)	120	35	151	20.0
Neuilly	128	32	172	11.6
Taaf (*South Wales*)	140	35	244	8.0
Turin	147	18	293	10.2
London	152	38	215	14.0
Chester	200	42	349	8.6

CAST-STEEL ARCH.

NAME OF THE ARCH.	SPAN. Feet.	FACTOR OF SAFETY.
Illinois and St. Louis Bridge	515	6+‡

* Irwin on *Iron Bridges and Roofs.*
† Cresy's *Encyclopædia.*
‡ *Report of the Engineer*, p. 33.

SUSPENSION BRIDGES.

NAME OF THE BRIDGE.	Span in feet.	Strain in tons per square inch. From Bridge.	Strain in tons per square inch. Bridge and Load.	Factor of safety.
Menai	580	4.21	8.00	3.9*
Hammersmith	422½	5.38	9.36	3.3*
Pesth	666	5.01	8.11	3.9*
Chelsea	384	4.36	8.07	3 9*
Clifton	702¼	2.90	5.03	6.4*
Niagara	821	6.70	8.40	5.3†
Suspension Aqueduct, Pittsburgh, Pa. 7 spans each,	160	....	....	4.0
Cincinnati Bridge ‡	1,057	9.1	11.7	6.2
East River	1,600§	....	....	6.0
Highland (*proposed*)	1,665‖	....	....	6.0

TUBULAR BRIDGES.

NAME OF THE BRIDGE.	SPAN. Feet.	FOR WEIGHT OF BRIDGE AND LOAD. Tension. Tons.	Compression. Tons.
Conway	400	6.85	5.03
Britannia (Central span)	460	3.00	....
Penrith (Tubular Girder)	...	4.75	4.25

* Tensile strength, 70,000 lbs. per square inch.
† Tensile strength, 100,000 lbs. per square inch.
‡ *Report of the Chief Engineer*, J. A. Roebling.
§ *Engineer's Report.* Suspending ties, factor of safety, 8.
‖ *Jour. Frank. Inst.*, vol. LXXXVII., p. 165.

STONE FOUNDATIONS.

	FACTOR OF SAFETY.
Pillars of the Dome of St. Peter's (*Rome*)..............	16
" " St. Paul's (*London*)..............	14
" " St. Geneviève (*Paris*)............	7.6
Pillars of the Church Toussaint (*Angers*)*...............	10
Merchants' Shot Tower (*Baltimore*)....................	4.8
Lower courses of Britannia Bridge.....................	31
Lower courses of the Piers of Neuilly Bridge (*Paris*)....	15.8
Foundation of St. Charles' Bridge (*Missouri*)	12 to 14
Foundations of East River Bridge †....................	10 to 20

199. PROOF LOAD.—The proof load is a trial load. It is intended as a practical test of a structure.

It generally exceeds the greatest load that it is ever intended to put upon the structure when in actual service.

According to the principles which have been discussed in the preceding pages, it is evidently better for the structure, and should be more satisfactory, to apply a moderate *proof load* for a long time than an *excessive* one for a short time.

* Strength of Materials, *J. K. Whildin*, p. 23.

† "In the stone work the pressures vary from 8 to 26 tons per square foot. Stone used is granite, selected samples of which have borne a crushing strain of 600 tons per square foot. Some will not bear over 100 tons per square foot. The general average is necessarily much less than that of the best specimens."—*Statement of the Chief Engineer, Washington A. Roebling.*

APPENDIX I.

TIMBER AND ITS PRESERVATION.

(The author is permitted to introduce here an abstract of portions of lectures delivered to his classes by Professor R. H. Thurston, of the Steven's Institute of Technology, on the above subject.)

The term *timber*, in the trade, is applied to logs cut from trees which are above six or eight inches in diameter.

Before felling, it is called *standing timber;* when first cut, it is called *rough timber*, and after it has been sawn, it is called *converted timber*, and is also known as *sided timber*, *joist*, *plank* or *board*, according to dimensions.

Wood is either *soft wood* or *hard wood*. The first class includes the wood of all coniferous trees, as the pines, and of a few others, as, for example, white birch. The second class includes the wood of all other timber-producing trees.

The soft woods generally contain turpentine and pitch, and are usually of rapid growth, straight grained, of slight density, quite uniform in texture, and comparatively free from knots. They have but little lateral adhesion of fibre, and are easily worked.

The hard woods are denser, heavier and stronger, less easily sawn, split, or cut, and are more liable to warp and to crack than are the soft woods. They usually excel in durability, and, in some cases, are very tough and elastic.

Good timber has the following characteristics:—

The heaviest is usually the strongest and most durable.

That which has least sap or resin is the best.

The freshly-cut surfaces are firm and smooth, and the shavings are translucent, and should nowhere appear chalky or roughened, that being the first indication of decay.

The annual rings should be closely packed, and the cellular tissue of the medullary rays should be hard and dense.

The tissues should cohere firmly, and, when sawn, there should be no wool-like fibre clogging the saw-teeth.

In general, the darker the color the stronger and more durable the wood.

Climate and Soil greatly affect the value of timber. Generally the strongest varieties of wood come from tropical climates, but the best examples of any one variety, are usually from the colder portion of the range of country in which it abounds.

Timber of slow growth, in situations protected from violent winds, cut at the right time of year, and properly seasoned, is free from cracks and shakes.

Cup shakes are produced by the wrenching of the tree by winds, and are cracks separating one layer from another. Timber thus injured is sometimes called "rolled timber." Longitudinal cracks are produced by heavy winds also, and by too rapid seasoning; in the latter case they are called *seasoning cracks*, in the former, *wind shakes*. Frost, in cold climates, sometimes produces this kind of injury.

Timber decays in two quite different ways, the causes of decay being, however, the same in both cases, namely, fermentation and putrefaction.

Dryness is the best preventive of decay of timber used in general construction, and wood kept dry has been found to last several centuries. Still, it finally becomes brittle and weakened, and may ultimately give way under a light load.

Water seems to act as a preservative, and some kinds of timber, constantly immersed in water not in motion, may endure for an indefinite period. The first effect of water is to dissolve out soluble matters, leaving the woody fibre or *lignin* uninjured, except, perhaps, very slightly by oxygen in solution in the water. This oxygen being exhausted, however, no further action occurs, unless a fresh supply of air-laden water displaces that originally in contact with the wood.

Alternation of moisture and dryness induces rapid decay. This takes place partly by solution and removal of a portion of the substance at each moistening, and partly by the action of oxygen dissolved in the water, a fresh supply of dissolved oxygen being furnished at each repetition of the moistening. The intermittent process serves to render the oxydation more rapid than it would otherwise be.

Continued dampness in a warm atmosphere is most favorable to fermentation, and, consequently, to rapid decay. This putrefaction of woody fibre is known as "rot," among those who use timber. The presence of water is necessary, as well as that of air, to the rapid progress of this chemical change, although the oxygen which is essential may sometimes be obtained from some source other than the atmosphere.

The products of this decomposition are, as in cases of rapid combustion of wood, carbonic acid and water.

Sap wood is more perishable than heart wood, in consequence of the presence of saccharine and other matters having a peculiar tendency to fermentation. It is in consequence of this fact that the complete removal of the sap by seasoning is necessary.

Lime, by its tendency to abstract carbon, which, uniting with oxygen, combines with lime to form the carbonate, hastens the rotting of wood wherever it is damp. Dry lime and the carbonate do not have this effect.

"*Wet rot*" and "*dry rot*" are the two forms in which the decay of timber exhibits itself.

Wet rot occurs in any portion of the wood, if damp, and attacks the heart wood of standing timber. Dry rot is usually produced by the want of circulation of air, and by high temperature, where the timber has not been well seasoned.

The most rapidly growing trees are most subject to decay, and those grown in sheltered localities are more liable to rot than those in exposed situations. Of soft timber, that containing most turpentine is least liable to rot.

Wood-work imbedded in damp plaster, and unseasoned timber covered with a coating of paint, are subject to dry rot, and are apt to decay early, in consequence of the confinement of air and moisture within their pores. Anything which absorbs moisture and confines it in contact with wood, is likely to accelerate decay.

Marine animals frequently attack timber immersed in salt water, as the bottoms of vessels, piles, etc.

The *Teredo Navalis*, commonly known as the ship-worm, converts the wood which it enters into a perfect honey-comb. It enters the wood when very small, and there increases in size, and enlarges its chambers correspondingly, until it sometimes makes borings an inch in diameter and several feet long. Soft woods are very rapidly destroyed by it, and the hardest woods are not safe against its attacks.

The *Limnoria Terebrans* is a smaller creature than the Teredo, shaped somewhat like a wood-louse, and is rather more than an eighth of an inch long. It is very destructive, cutting out the wood along the annual rings.

There are several other marine animals which attack timber, and it is usually necessary to protect it when immersed in salt water by sheathing with copper, as ships are protected, or otherwise covering it with a coating impenetrable by these animals. Some kinds of timber are much less liable to this kind of injury than others. The East Indian teak is said never to be attacked by either of these creatures, and live oak is comparatively little injured by them.

PRESERVATION OF TIMBER.

The causes of decay in timber have already been stated, and the process of decay has been described.

The problem of preserving timber from decay is evidently fully stated when it is said that the object to be obtained is the prevention of oxidation.

Timber which has been thoroughly seasoned by the methods already described, and which is perfectly dry, may be preserved by external applications. Under other circumstances, internal application of various solutions must be resorted to.

Paints and Varnishes are used for the protection and preservation of timber by external treatment.

They form a coating upon the surface, which resists the wearing action of the weather, and prevents the entrance into the pores of the wood of either moisture or corroding gases.

Should the wood not have been previously well-seasoned, however, paint only hastens decay by confining the moisture and hastening the fermentation of the putrescible matter remaining in the wood.

Wood-work exposed to the weather should be repainted at intervals of four or five years in our climate.

The wood-work supporting the floors of bridges, and timber in damp situations, as in wheel pits, is sometimes coated with coal tar, prepared for use by boiling, and by the addition of a small quantity of chalk to give it body.

Boiling linseed oil, pitch and vegetable tar, applied hot, are not unfrequently used as external applications, and are found to be very effective preservatives.

Sulphate of Iron, in oil, has also been found to make a useful paint.

Charring the surface of well-seasoned timber is found to considerably increase its durability, and is frequently adopted for the preservation of those portions of fence-posts which are buried in the ground.

An external application of *Silicate of Sodium* has been advised by Abel for seasoned timber. It is said to form a hard, and very durable coating upon the surface, and to act effectively as a preservative against fire as well as against decay.

The solution is laid on with alternate coats of limewash. Two or three applications of the silicate of sodium are required to form each coat.

Sulphates of iron, and of copper, the chloride of mercury, common salt, and other solutions, are occasionally used for external washes, but the common oil paints, are by far the most usually applied. Their durability is increased by sprinkling liberally with sand when circumstances will permit. In timber protected by external treatment special care is required to fill all cracks.

The preservation of timber, either seasoned or unseasoned, by saturation with antiseptic solutions, has become a matter of such great importance as to have attracted much attention. Many processes have been tried and recommended, but none are practised in this country, and very few are practised at all.

A few seem to be effective but costly, many are of temporary benefit, and others, while seeming to be useful at first, are actually injurious, ultimately destroying the timber which they are intended to preserve.

The external applications above described, are of no value in defending the timber against the attacks of wood-boring insects. Sheathing the timber in metal, and one or two methods of saturation, are apparently the only reliable expedients.

Of the processes of preservation of wood by saturation, Kyan's consists in the injection of the *bichloride of mercury* (corrosive sublimate); Burnett used the *chloride of zinc;* Boucherie employed the *pyrolignite of iron;* Margery the *sulphate of copper;* Bethell saturated his timber with *creosote*, or "dead oil" from gas works; Beer used a solution of borax.

The metallic salts owe their antiseptic property to the fact that they produce coagulation of the albumen, which is the fermentable and perishable part of timber.

The use of metallic salts was proposed nearly a century ago, but the first practical applications were made about forty-five years ago.

"Kyanizing" was suggested by Sir Humphrey Davy, some ten years before the process was patented by Sir R. H. Kyan, in England, in 1832.

The solution used consisted of one pound of the chloride in four gallons of water.

Timber thoroughly impregnated with the salt has great durability, but the general adoption of this process is precluded by the cost of materials. A hundred pounds of timber absorbed one and a half pounds of corrosive sublimate. Where it is brought in contact with iron, it produces corrosion, and its application is thus rendered still less frequently permissible.

Kyanized timber was used to some extent in Great Britain when first proposed, and in the United States. Among other constructions of timber thus prepared, may be mentioned the aqueduct of the Alexandria Canal crossing the Potomac River at Georgetown.

"*Burnettizing*" was proposed by Sir Wm. Burnett in 1838, and has been quite largely practised for special purpose.

The chloride of zinc, in the proportion of one pound dissolved in ten pounds of water, is forced into the pores of the wood under a pressure of from one hundred to one hundred and twenty-five pounds to the square inch. Burnett's method was, originally, simple immersion in the solution two or three weeks.

An establishment was organized at Lowell, Mass., in 1856, in which Burnettizing under pressure was practised, and, subsequently, several railroad companies adopted this method and process, and erected Burnettizing works.

The cost of preserving timber by this method, including interest on capital and all other expenses, ranges from five to seven dollars per thousand feet, board measure.

The process is not, however, believed to afford as perfect protection as the more expensive method of Kyanizing.

The Bethel Process was also patented in England in 1838, and its cheapness and effectiveness have given it a considerable commercial success both in Europe and the United States.

It consists in the saturation of the wood with bituminous substances obtained by the distillation of coal-tar. Like the metallic salts, these substances produce coagulation of the albumen, and thus destroy the tendency to fermentation.

Timber thus prepared is rendered very durable, and the process is comparatively inexpensive.

Its use has, however, been given up in some instances, after extended trial, on the ground that the increase in durability was not sufficient to compensate for the expense.

Each cubic foot of timber, under a pressure of one hundred and fifty pounds per square inch, absorbs in twelve hours from eight to twelve pounds of the creosote, or dead oil. The smaller amount is the allowance advised for railroad cross-ties. Hard wood absorbs least.

The strength of timber, preserved by this method, is unimpaired, and it requires no painting, although with dry timber a superficial coating of coal-tar is sometimes added.

This process has special advantages when the timber is exposed to alternations of dryness and moisture, and therefore liable to wet-rot. The dead oil fills the pores completely, coagulates all albumen, absorbs all oxygen that may exist free in the wood, and by its poisonous qualities it acts as a protection against the attacks of insects.

The Seely and Robbins processes are American modifications of the Bethel process.

The *Seely* process consists in subjecting the wood to a temperature between 212° and 250° Fahr., in a bath of creosote oil, for a sufficient length of time to expel all moisture. When all water is thus expelled, the pores contain only steam. The hot oil is then quickly replaced by a bath of cold dead oil. The steam in the pores of the wood is thus condensed, and a vacuum is formed, into which the oil is forced by atmospheric pressure and by capillary attraction.

From six to twelve pounds of creosote oil to the cubic foot of wood is expended in this process. The amount is dependent upon the use to which the wood is to be put. For piles or other timber exposed to the depredations of worms, twelve pounds is used.

An impregnation of ten pounds to the cubic foot costs 25 cents. For piles the usual charge is 30 cents per cubic foot.*

For work in wheel pits and under foundation, at least ten pounds per cubic foot should be used.

The *Robbins* process consists in treating wood with coal-tar or oleaginous substances in the form of vapor. The wood is placed in an air-tight iron chamber, connected with which is a still, or retort, heated by a furnace.

When heat is applied the vapor of naphtha is generated at a temperature of 250° to 300°, the creosote oil vapor at 360° to 400°, and the heavier tar oils at 500° to 600° Fahr. The wood is thus exposed for from six to twelve hours.

By this process it would seem impossible to charge the wood with more than a fraction of the amount of carbolic acid, and of other component parts of coal-tar, expended in the Seely process. The latter process is decidedly an improvement on the process of Bethel.

Even marine insects usually avoid creosoted timber, and wood so prepared is therefore used to a considerable extent for submarine work.

In tropical countries it does not, however, protect entirely against the attacks of the white ant.

The antiseptic element of dead oil is supposed to be its carbolic acid, which is estimated by Prof. Letheby at from ½ to 6 per cent. of the whole. The cost of creosoting 1,000 cubic feet, board measure, of oak or of spruce fir, has been given as from five to eight dollars.

Dr. Boucherie patented, in 1839, an ingenious and inexpensive method of saturation. This process received much attention, and was practised with considerable success.

The timber, freshly cut, and with its terminal foliage still remaining, was set, either vertically or horizontally, with the foot immersed in a vat containing the antiseptic solution.

The circulation continuing in the trunk of the tree, the sap becomes ejected, and its place is taken by the preservative solution, which is thus thoroughly distributed throughout the fibre. Growing trees were also treated by the injection of the liquid into their trunks.

Where logs, deprived of foliage and branches, were to be saturated, they were placed on end, and a waterproof bag, or a tank, containing the solution used was mounted above it, the liquid being thus forced downward through the stick by hydrostatic pressure, driving the sap before it and out of the lower end.

The antiseptic proposed by Dr. Boucherie was crude *pyrolignite of iron*. His process of saturation was largely used with other preservatives also, and his invention of the saturating process seems to have been more generally appreciated than his introduction of a cheap antiseptic. Where it can be conveniently applied it is exceedingly efficient.

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* 1874.

Numerous and elaborate experiments were tried by Dr. Boucherie, in which the action of pyrolignite of iron was carefully noted.

He found that one-fiftieth the weight of the green wood was a sufficiently large proportion of the antiseptic to ensure preservation.

The hardness of the wood was stated to be double by the use of the pyrolignite.

Solutions of deliquescent salts were applied by Dr. Boucherie in the way described, and were found, in the case of chloride of lime and some others, to increase the flexibility of timber.

He therefore proposed the use of such solutions, with the addition of one-fifth their quantity of pyrolignite of iron, when it was desired that the wood should retain its moisture, and its flexibility and elasticity. The same inventor proposed, as a cheap substitute for these solutions, the stagnant water of salt marshes.

Such preparation, it was claimed, also prevented the warping and splitting of wood, which is a frequent consequence of rapid drying, and yet seasoning was said to be expedited by its use. In this case the solutions were weak, and the wood could be afterward painted over without difficulty.

The process was applied by the inventor to the saturation of timber with earthy chlorides as a protection against fire. These salts, fusing upon the surface of the wood, on the application of heat, rendered it quite incombustible.

Wood was dyed with both mineral and vegetable colors by Dr. Boucherie, and the application of the usual methods of producing "fast" colors, by the introduction of dye and mordant successively, was thus made practicable. Wood was treated with odorous solutions to give them fragrance, and with resinous matter to make them waterproof.

The French Government, after receiving favorable reports from the Commission of Engineers appointed to examine into the merits of the process, finally conferred upon the inventor the Great Gold Medal of Honor. Subsequently a money award was made him, and he surrendered his patent, which thus became public property.

"Beerizing" consists in the saturation of the timber, by any convenient process, with a solution of borax. This is claimed to dissolve the albumen, and the solution may be allowed to remain, the borax having antiseptic properties. Or it may be washed out, and the wood, being then dried, is stated to become more thoroughly seasoned and durable than it can be made by the ordinary processes of seasoning.

Margery's process of saturation with sulphate of copper has been found very effective in some instances.

It was applied, by the Boucherie method, to telegraph poles and to railroad cross-ties many years ago in France, with perfect success, as a preservative against decay. Major Sankey found it equally efficient in India, more recently, as a protective against the attacks of the white ant and of other insects.

The latter used a solution of one pound of the salt in four gallons of water. The timber was steeped in the solution two or two and a half days for each inch in thickness.

A simple coating of boiled linseed oil, thickened with powdered charcoal, has been in some cases found a very economical and efficient preservation of timber.

The preservation of timber is of daily increasing importance, not only as a matter of ordinary economical policy, but because the rapid destruction of forests is continually rendering timber more scarce and more costly. It cannot be many years before it will become a matter of such vital necessity to preserve our forest trees, that legislation will inevitably aid in increasing the market value of timber, by forbidding its wholesale destruction.

The substitution of iron for wood, in construction, is proceeding so rapidly that it will afford some relief, but this will nevertheless remain a matter of exceptional importance.

H. W. Lewis, writing in 1866, says: "Allowing only 2,000 sleepers to a mile, at a cost of fifty cents each, and admitting the average life of American sleepers (cross-ties) to be only seven years, and that it costs ten cents to treat each tie in some way so as to make it last fourteen years, then the saving at the end of seven years is $600 per mile.

"There are in the United States about 45,000* miles of railroad, and hence, if the above conditions could be realized on all of them, the annual saving would be about $3,400,000."

Our railroads are now vastly extended, and the other uses to which timber is put, which allow of the application of preservative processes, are many and important. The value to the country and to the world of effective and cheap processes of preservation cannot be estimated.

The following tabular statement of experimental results obtained by various processes is given in a report to the Board of Public Works of the District of Columbia, as derived from an examination of those methods by Drs. B. F. Craig and W. C. Tilden of the U. S. Army, in the laboratory of the Surgeon-general's Office at Washington.

* The length of railroad in 1873 was reported to be 70,000 miles.

EXPERIMENTAL RESULTS.

Number and size of each block as received.	1 Absorptive power (per cent. of distilled water absorbed in 48 hours).	2 Per cent. of solution and mechanical removal in the preceding experiment.	3 Per cent. of matters dissolved by the proper solvents from the centre of each block.	4 Per cent. dissolved from the superficial portions of each block.	5 Ash, per cent. at centre.	6 Ash, per cent. near the surface.	7 Apparent specific gravity at centre of each block.	8 Apparent specific gravity of superficial portions of each block.	9 Weight of one cubic foot of density of column 7.	10 Weight of one cubic foot of density expressed in column 8.	11 Reaction of the Wood.
1. 8x3x6 in.	per cent 24.29	per cent 1.00	per cent 4.55	per cent 33.8	per cent 1.72	per cent 13.6	0.412	0.568	lbs. 25.1	lbs. 35.5	Acid.
2. 8x8x4½ in.	47.90	0.21	2.84	7.7	1.86	6.2	0.445	0.472	27.6	29.5	Acid.
3. 2x6x1 in.	5.08	0.43	7.7	5.08	0.33	0.4	1.116	1.007	69.7	62.9	Acid.
4. 4x6x2 in.	7.25	0.39	85.4	40.00	2.1		0.707	0.700	44.1	43.7	Faintly Acid.
5. 4x4½x2 in.	46.64	1.01	2.64	9.55	1.63	7.2	0.619	0.6706	38.6	41.9	Faintly Acid.
6. 10x6x4 in.	3.32	0.16	centre & surface. 17.		0.35		0.5608	0.650	35.0	40.6	Faintly Acid.
7. 9x4x1 in.	8.46	0.33	centre & surface. 22.8		0.45		0.845		52.8		Faintly Acid.
8. 6x6x3 in.	97.58	0.46	1.55	3.3	1.46	5.4	0.484	0.498	40.2	41.1	Faintly Acid.
9. 7x6½x5 in.	31.86	0.26	1.16	3.45	1.45	1.60	0.4609	0.439	28.8	27.4	Acid.
10. 12x6x3 in.	22.80	1.00	1.96	12.25	0.52		0.4202	0.445	26.2	27.8	Faintly Acid.
11. 12x6x4 in.	16.15	0.5	14.83	23.7	0.38		0.7005	0.833	43.7	52.0	Faintly Acid.
12. 12x7x4 in.	13.54	0.10	16.2	17.6	0.29		0.601	0.633	37.5	39.5	Faintly Acid.
13. 4x5x3 in.	2.88	0.13	55.07	41.4	0.36		1.095	1.064	68.4	66.5	Acid.
14. 6x3x2 in.	21.27	5.44	81.3	73.5	21.9	24.8	0.455		28.4		Acid.
15. 8x6x3 in.	23.19	0.23	10.8	13.4	0.52		0.6004	0.617	37.5	38.5	Faintly Acid.
16. 10x6x3 in.	29.11	0.59	1.91	7.5	1.8	3.8	0.449	0.554	28.0	34.6	Acid.
17. 10x6x3 in.	31.98	9.54	9.17	67.0	6.74	38.8	0.449	0.541	28.0	[illegible]	Neutral.
18. 5x4x3 in.	28.93	0.16					0.493		30.8		
19. 5x4x3 in.	62.74	0.64									Acid.
20. 10x6x3 in.	78.10	0.21					0.429		26.8		
21.			1.57	4.7	1.63		0.519				
22. Small piece from a pavement in N. Y., laid three years			85.				0.407		25.4 (very dry.)		

MINERAL AND METALLIC PROCESSES.

NAME AND NUMBER.	BRIEF OF CLAIM.
Burnettized Spruce. Two specimens. Nos. 2 and 9. (No. 2 is a piece of railroad tie, said to have been buried sixteen years.)	"The Burnettizing process consists in placing the wood in large wrought-iron cylinders; then extracting the air and sap contained in the pores of the wood by a vacuum. The solution of chloride of zinc is then allowed to run in, and a pressure of from 150 to 160 lbs. per square inch applied to force the zinc into the pores." Perfect coagulation of albumen and entire indestructibility by wet or dry rot are claimed.
A. B. Tripler's Arsenic Process. One specimen. No. 14.	Saturation of blocks composing a wooden pavement with chloride of arsenic, or arsenic and chloride of sodium, and coating them on their upper surface with a resinous or tarry waterproof composition. Also, the interposition of an antiseptic compound between the blocks and the earth, by either soaking the foundation planks or mixing the antiseptic with the sand.
The Samuel's Process. Nos. 8 and 5 (?).	"Injecting into the pores of the wood, first, a solution of sulphate of iron, and afterwards a solution of common burnt lime, to render the wood in a high degree impervious to the influence of wet and dry rot, and the attacks of worms and other insects."
Thilmany's Process. One specimen. No. 16.	Saturation with sulphate of copper, followed by muriate of barytes, to form insoluble sulphate of barytes in the wood.
Process of Wirt and Hurdle. Specimens 18 (a) and 18 (b).	Charring the wood and covering the whole block with asphaltum.
Tait's Process. One specimen said to have been sent, marked No. 5. Analysis, however, places this block with No. 8, as a sample of the Samuel's process.	"Charging or saturating the pores of the wood with a concentrated solution of bi-sulphite of lime or baryta, the same being rendered soluble by excess of sulphuric acid gas, under pressure or by refrigeration, and being made insoluble as a neutral sulphate when the pressure or excess of gas is removed."
Thomas Taylor's Process. Two specimens. Nos. 19 & 21.	Uses a solution of sulphide of calcium in pyroligneous acid for the impregnation of the wood; or, uses sulphide of calcium first, and follows it with pyroligneous acid. Claims a deposit of pure sulphur through whole block.
Thompson & Co.'s Process. (Arsenic.) No. 17.	No description or explanation of process furnished. Claims "indestructibility" and "non-inflammability."

REMARKS.

Fibre of blocks weak and brittle; color grayish.

Absorptive power greater than that of natural wood. All of the zinc easily removed by acidulated water. Evidences of the partial decomposition of the zinc chloride observed. Uneven character of impregnation shown both by microscopic examination and by unequal percentage of mineral matters removed by acidulated water from centre and near surface. (See Columns 3 and 4 of Experimental Results.)

Size of specimen very small, yet the impregnation uneven. (See Columns 3 and 4.) Quantities of soluble salts very large. No arsenic found, though its use is claimed. The resinous covering designed to protect the top of each block is worthless for the purpose, for obvious reasons, chiefly its brittleness.

Absorptive power high.

Absorptive power too high for representation on the chart. Wood brittle and readily splintered. Impregnation very unequal. The water used for Experiment No. 1 (absorptive), was filled with threads of fungi after standing forty-eight hours, showing that it is doubtful if even dry rot can be prevented by this process.

Saturation very uneven. Absorptive power high.

Block contains soluble salts of copper removable by washing.

Process inapplicable to unseasoned timber. The asphalt covering melts and flows at 60° to 70° F. When cold and brittle, the wear of the pavement will remove it, leaving each block as a porous cup for the reception of water which cannot drain through it. Process not considered worth particular investigation.

It is doubtful if any specimen was received. No. 5 resembles the "ironized" blocks. If claimed as a sample of the Tait process, the same memoranda are made upon it as upon No. 8.

The claims of this process are not substantiated.

No sulphur uncombined found in any part of blocks submitted.

About nine-tenths the whole bulk of each block possessed every property of seasoned white pine untreated by any method whatever.

Between three and four per cent. sulphate of lime found in superficial portions.

An arsenic process. Absorption power high. Specimen is cottonwood.

Saturation extremely uneven. Solubility of saline ingredients complete.

CREOSOTE OIL AND RESIN PROCESS.

Laboratory Surgeon-General's Office.

Name and Number.	Brief of Claim.	Remarks.
Waterbury's Process. One Specimen. No. 1.	Treats wood in closed cylinder with steam to vaporize sap; then introduces a solution of *common salt*, followed by *dead oil*, *creosote oil*, or equivalent. Claims complete impregnation by both substances.	Absorption figures high. Saturation by solution of common salt is only partial. Columns 3 and 4 show a very uneven penetration by "*dead oil.*" Water dissolves out all the salt used. Columns 5 and 6 show the uneven distribution of mineral matters.
Thomas' Process. Two Specimens. Nos. 3 and 13.	Two small blocks, 2x6x1 and 4x5x8 in., were sent without explanation or name; the substance used for impregnation is "resin oil."	Absorption power low. Physical condition of specimens very bad. Saturating material easily soluble in alkaline fluids. The strength of wood in these samples stands at a minimum, especially its transverse and crushing strength.
Seely's Process. Five Specimens. **Pelton's Apparatus** for applying Seely Process. Nos. 4, 6, 12, 22.	Immersion of wood in a bath of *creosote oil* or other suitable material, heated to about 250° F., until it is evident that air and moisture are eliminated; then substituting for the hot bath, one at as low a temperature as allows perfect fluidity, the liquor being also *dead oil.* Claims that the pores of the wood are in a vacuous condition as it cools, and that the impregnating material readily fills them by capillary action and atmospheric pressure.	Average absorption power very low. Saturation thorough and very uniform. (See Columns 3, 4, 9, and 10.) Solid hydrocarbons present within the cells. Condition of fibre uninjured.
Robbins' Process. Two Specimens. Nos. 10 and 20.	Claims to impregnate wood with *light and heavy oils of tar*, by exposure in a chamber connected with a *retort* or *still* in which the oils are *vaporized;* states that naphthalin and other solid hydrocarbon bodies are distilled over into the wood and condensed in its pores; also that all moisture is driven out and the albumen coagulated.	Absorption power very high. Percentage of liquid hydrocarbons very low in all portions of block except the outer. No solid hydrocarbons observed, even on surface (naphthalin, etc.). Condition of wood shows injury from heat. Specimens are evidently suited for exposure to *dry air* only, under which circumstances the protection is sufficient.
Detwiler and Van Gilder Process. No. 11.	Impregnation of wood by *resin* dissolved in *naphtha*, under pressure, and at high temperature.	Saturation uneven. (Columns 3 and 4, also 9 and 10.) Absorption power quite high.
U. S. Antiseptic Wood Co.'s Process. Constant and Smith Patents. No. 15.	*Dries* or *seasons* wood by *hot air;* preserves it (when desired) by generating "*smoky vapors*" in a retort, the same being allowed to penetrate the wood and to condense within its pores.	The same remarks made under Nos. 10 and 20 (Robbins' process) apply to this specimen, with the difference that the experimental results show the Robbins' process to be very much superior to this which presents identical claims.

APPENDIX II.

(The *numbers* of the articles correspond with those in the text.)

19. To integrate $\frac{d^2x}{dt^2} = P - \frac{EK}{l}x$; multiply both members by dx, and the first member becomes $\frac{dxd^2x}{dt^2}$. Let $dx = z$, then $d^2x = dz$, and the expression becomes $\frac{zdz}{dt^2}$. t being the independent variable, dt is constant, and hence the integral is $\frac{z^2}{2dt^2} = \frac{dx^2}{2dt^2}$. The other reductions are evident.

The *resilience* (or spring of the bar) is the work of elongating it to the limit of proof strain. Equation (7) of the text gives

$$U = \frac{EK\lambda^2}{2l} = \frac{EK}{2l} \cdot \frac{P^2l^2}{E^2K^2} = \frac{P^2}{EK^2} \cdot \tfrac{1}{2}Kl.$$

$P \div K$ is the stress on a unit of surface. The quantity $\frac{P^2}{EK^2}$, is called the modulus of resilience.

22. The strength of metals is referred to the original section, but before rupture takes place the section is considerably reduced. See the table in Article 35, and the example in Fig. 116. In some cases it may be reduced to $\frac{1}{3}$ of its original section before it ruptures.

32. At the bottom of p. 40 there is a remark which, in the light of more recent experiments, is probably incorrect. The probable effect of the tension was to relieve the iron of internal strains, and by leaving it for a time without strain, and permitting it to cool, all the elements may come to a condition of maximum resistance.

96. The moment of inertia of a surface is the sum of all the products obtained by multiplying each elementary area by the square of its distance from an axis.

The moment of inertia of a volume is defined in a similar way. Similarly for a weight, or for a mass.

MOMENT OF INERTIA OF A SURFACE.

Let I = the moment of inertia about an axis;
dA = the elementary area; and
y = an ordinate from the axis to the element.
Then

$$I = \int y^2 dA \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

If the axes are rectangular $dA = dydx$,

$$\therefore I = \iint y^2 dydx$$

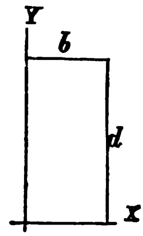

Fig. 123.

Example.—Required the moment of inertia of a rectangle when the axis coincides with one end.
Here

$$I = \int_0^d \int_0^b y^2 dydx = \tfrac{1}{3}bd^3$$

FORMULA OF REDUCTION.

Let I_0 = the moment of inertia about an axis through the centre;
I = the moment about a parallel axis;
D = the distance between the axes (Fig. 83*b*); and
y_0 = the ordinate from the axis through the centre to an element.

$$\text{Then } I = \int y^2 dA = \int (y_0 - D)^2 \, dA = \int y_0^2 \, dA - \int 2Dy_0 dA + \int D^2 dA$$

$$= I_0 + D^2 A,$$

for $2D\int y_0 dA = 0$ since the axis passes through the centre. (See foot note, p. 88.)

That is, *the moment of inertia about any axis equals the moment about a parallel axis through the centre, plus the area of the section multiplied by the square of the distances between the axes.*

From the preceding equation we have

$$I_0 = I - AD^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Example.—Required the moment of inertia of a rectangle about an axis through the centre and parallel to one end.

Here $A = bd$, $D^2 = \frac{1}{4}d^2$, and $I = \frac{1}{3}bd^3$ as found above; hence

$$I_0 = \tfrac{1}{3}bd^3 - \tfrac{1}{4}bd^3 = \tfrac{1}{12}bd^3;$$

as given in Eq. (51) of the text.

If the *moment axis* is perpendicular to the surface, we let

ρ = the variable distance of any element from the axis;
ϕ = the variable angle;
$dA = \rho d\phi d\rho$ (see Fig. 78); and
I_p = the *polar* moment of inertia.

$$\therefore I_p = \int \rho^2 dA$$

But $\rho^2 = x^2 + y^2$

$$\therefore I_p = \int x^2 dA + \int y^2 dA = I_y + I_x \dots\dots\dots\dots\dots (4)$$

in which I_y is the moment about y and I_x the moment about x. If $I_x = I_y$, we have

$$I_p = 2I_x.$$

The polar moment of inertia of a circle is Eq. (199) of the text,

$$I_p = \tfrac{1}{2}\pi r^4 = 2I_x$$

$$\therefore I_x = \tfrac{1}{4}\pi r^4 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (5)$$

which is the moment about a diameter, as given in Eq. (52).

TO FIND THE RELATION BETWEEN THE MOMENTS OF INERTIA ABOUT DIFFERENT AXIS HAVING THE SAME ORIGIN.

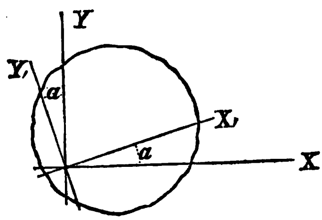

FIG. 124.

Let x and y be rectangular axes,

x_1 and y_1, also rectangular, having the same origin;

a = the angle between x and x_1;

I_x = the moment of inertia about the axis x, similarly for I_y, I_{x_1} and I_{y_1};

$B = \int xy dA$; and

$B_1 = \int x_1 y_1 dA.$

For the transformation of coördinates, we have

$$x_1 = x \cos a - y \sin a$$
$$y_1 = x \sin a + y \cos a$$
$$x_1^2 + y_1^2 = x^2 + y^2.$$

Also

$$dA = dxdy = dx_1 dy_1.$$

Hence,

$$\left.\begin{aligned} I_{x_1} &= \int y_1^2\, dA = I_x \cos^2 a + I_y \sin^2 a - 2B \cos a \sin a \\ I_{y_1} &= I_x \sin^2 a + I_y \cos^2 a + 2B \cos a \sin a \\ B_1 &= (I_x - I_y) \cos a \sin a + B(\cos^2 a - \sin^2 a) \\ \therefore I_{x_1} + I_{y_1} &= I_x + I_y = I_p \end{aligned}\right\} \dots (6)$$

which is an *isotropic* function; since the sum of the moments relatively to a pair of rectangular axes equals the sum of the moments relatively to any other two rectangular axes having the some origin; or, in other words, the sum of the moments of inertia relatively to a pair of rectangular axes is constant.

From the first of (6) we have by differentiation and placing equal to zero for a maximum or minimum,

$$\frac{dI_{x1}}{da} = (I_x - I_y)\cos a \sin a - B(\cos^2 a - \sin^2 a) = 0$$

Similarly,

$$\frac{dI_{y1}}{da} = -(I_x - I_y)\cos a \sin a + B(\cos^2 a - \sin^2 a) = 0$$

$$\therefore B_1 = 0.$$

From the first or second of these we have

$$\frac{2B}{I_x - I_y} = \frac{2 \cos a \sin a}{\cos^2 a - \sin^2 a} = \tan 2a.$$

It may be shown by the ordinary tests that when I_{x1} is a maximum, that I_{y1} will be a minimum, and the reverse; hence *there is always a pair of rectangular axes in reference to one of which the moment of inertia is greater than for any other axis, and for the other it is less.*

These are called principal axes.

Thus, in case of a rectangle, if the axes are parallel to the sides and pass through the centre, we find

$$B = \int\int_{-\frac{1}{2}d}^{+\frac{1}{2}d} xydA = 0;$$

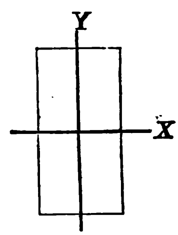

FIG. 125.

hence x and y are the maximum and minimum axes; and if $d > b$, $\frac{1}{12}bd^3$ is the maximum, and $\frac{1}{12}b^3d$ a minimum moment of inertia for all axes passing through the origin. In a similar way we find that if the origin be at any other point the axes must be parallel to the sides for maximum and minimum moments.

The preceding analysis gives the position of the axes for maximum and minimum moments, when the moments are known in reference to a pair of rectangular axes. But if the axes for maximum and minimum moments are known as I_x and I_y, then $B = 0$; and calling these I_x' and I_y', and Eqs. (6) become

$$\left.\begin{aligned} I_{x1} &= I_x' \cos^2 a + I_y' \sin^2 a \\ I_{y1} &= I_x' \sin^2 a + I_y' \cos^2 a \\ B_1 &= (I_x' - I_y') \cos a \sin a \end{aligned}\right\} \dots\dots\dots\dots\dots (7)$$

In the case of a square where the axes pass through the centre $I_x' = I_y'$

$$\begin{aligned} \therefore I_{x1} &= I_x' (\cos^2 a + \sin^2 a) = I_x' \\ I_{y1} &= I_y', \text{ and} \\ B_1 &= 0; \end{aligned}$$

hence the moment of inertia of a square is the same about all axes passing through its centre, and $= \frac{1}{12}b^4$. The same is true for all regular polygons, and hence for the circle.

Example.—To find the moment of inertia of a rectangle in reference to an axis which is inclined at an angle a to one side,

$$\text{We have } I_x' = \tfrac{1}{12}bd^3,\ I_y' = \tfrac{1}{12}b^3d$$

$$\left.\begin{aligned} \therefore I_{x1} &= \tfrac{1}{12}bd\,(d^2\cos^2 a + b^2\sin^2 a) \\ I_{y1} &= \tfrac{1}{12}bd\,(d^2\sin^2 a + b^2\cos^2 a) \end{aligned}\right\} \quad \text{.................(7)}$$

The latter value is the one given in Article 146.

The moment of inertia of a regular polygon may be found by dividing it into equal triangles, having their vertices at the centre, their bases being sides of the polygon, and find the moments of each in reference to an axis passing through their centre, and parallel to the main axis, which passes through the centre of the polygon, then reducing all of them in reference to the main axis. If R be the radius of the circumscribed circle, and r the radius of the inscribed circle, it may be found for any regular polygon, that

$$I = \tfrac{1}{24}A\,(R^2 + 2r^2) \quad \text{..................................(8)}$$

For the circle $R = r$

$$\therefore I = \tfrac{1}{4}\pi r^4,$$

as before found.

For the square $r = \frac{1}{2}b$, $R = \sqrt{2b^2} = b\sqrt{2}$, and $A = b^2$,

$$\therefore I = \tfrac{1}{12}b^4,$$

as before found.

96. The value of ρ (the radius of curvature) given in the latter part of this article, is taken directly from the Differential Calculus, but it may be easily found geometrically as follows :—

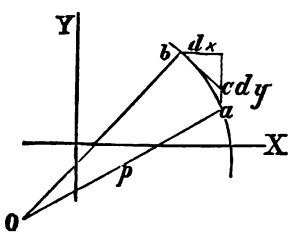

FIG. 126.

From two consecutive points erect normals, and where they meet at O will be the centre of the osculatory circle, and the normals will be sensibly equal, and will be the radius of curvature ρ. Let $d\phi$ be the angle between the normals; $ds =$ the arc $= \rho d\phi$.

$$\therefore d\phi = \frac{ds}{\rho}.$$

The coördinates of a in reference to b will be dx and dy. At b erect a perpendicular bc to Ob, and the angle abc will equal $d\phi$.

$$\frac{dy}{dx} = \text{tang } i.$$

Take the differential of this and we have

$$\frac{d^2y}{dx} = \frac{di}{\cos^2 i} = \frac{d\phi}{\frac{dx^2}{ds^2}}$$

$$\therefore d\phi = di = \frac{d^2y dx}{ds^2} = \frac{ds}{\rho}$$

$$\therefore \rho = \frac{ds^3}{d^2y dx} = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{d^2y dx}$$

99. For the integral of the expression $\frac{d^2y}{dx^2}$ see Article 19 of this Appendix.

114. Equations (130) and (131) are particular cases of the more general form

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = X = f(x) \quad \text{......(1)}$$

which may be integrated by La Grange's method of variation of parameters. (See Price's *Infinitesimal Calculus*, 1st Ed., Vol. II., pp. 479 to 481.)

First let $X = 0$, then

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0 \quad \text{......(2)}$$

Let $y = e^{mx}$ in which m is constant and e the base of the Naperian system of logarithms; then

$$\frac{dy}{dx} = me^{mx} \text{ and } \frac{d^2y}{dx^2} = m^2e^{mx} \quad \text{......(3)}$$

which in Eq. (2) becomes

$$m^2 + Am + B = 0 \quad \text{......(4)}$$

which is an equation of condition.

Solving gives

$$m = \tfrac{1}{2}A \pm \sqrt{\tfrac{1}{4}A^2 - B} = a \text{ and } b \text{ (say)} \quad \text{......(5)}$$

where a and b are the roots of Eq. (4). Hence the partial values of the first differential coefficient are

$$\left(\frac{dy}{dx}\right) = a\, e^{ax} \text{ and } \left(\frac{dy}{dx}\right) = be^{bx}\, dx.$$

In order that these should satisfy Eq. (1), a and b must be functions of x. Integrating each and introducing u_1 and u_2, which are functions of x instead of arbitrary constants, and we have

$$(y) = U_1e^{ax} \text{ and } (y) = U_2e^{bx}$$

$$\therefore y = U_1e^{ax} + U_2e^{bx} \quad \text{......(6)}$$

It now remains to find U_1 and U_2. Differentiate Eq. (6) and we find

$$\frac{dy}{dx} = U_1\, ae^{ax} + U_2 be^{bx} + e^{ax}\frac{dU_1}{dx} + e^{bx}\frac{dU_2}{dx} \dots\dots (7)$$

But such a relation may be established between U_1 and U_2 as that the sum of the last two terms shall be zero, or

$$e^{ax}\frac{dU_1}{dx} + e^{bx}\frac{dU_2}{dx} = 0 \dots\dots (8)$$

which is a second equation of condition. Hence Eq. (7) becomes

$$\frac{dy}{dx} = U_1 ae^{ax} + U_2 be^{bx} \dots\dots (9)$$

Differentiate again:—

$$\frac{d^2y}{dx^2} = U_1 a^2 e^{ax} + U_2 b^2 e^{bx} + ae^{ax}\frac{dU_1}{dx} + be^{bx}\frac{dU_2}{dx} \dots (10)$$

Substitute Equations (6), (9), and (10) in (1) and we have

$$U_1(a^2 + aA + B)\, e^{ax} + U_2(b^2 + bA + B)\, e^{bx} + ae^{ax}\frac{dU_1}{dx} + be^{bx}\frac{dU_2}{dx} = X.$$

But the first equation of condition, Eq. (4), reduces the first two terms to zero.

$$\therefore\ ae^{ax}\frac{dU_1}{dx} + be^{bx}\frac{dU_2}{dx} = X.$$

which, combined with Eq. (8), eliminating dU_1, and then dU_2, and we find

$$U_1 = C_1 + \frac{1}{a-b}\int Xe^{-ax}\,dx,$$

$$U_2 = C_2 - \frac{1}{a-b}\int Xe^{-bx}\,dx,$$

which substituted in Eq. (6) gives

$$y = C_1 e^{ax} + C_2 e^{bx} + \frac{e^{ax}}{a-b}\int Xe^{-ax}\,dx - \frac{e^{bx}}{a-b}\int Xe^{-bx}\,dx \dots (11)$$

By comparing Eq. (180) of Article 114 with Eq. (1) above gives

$$A = 0;\ B = -q^2 \text{ and } X = -p^2 x;$$

hence Eq. (4) gives

$$m = \pm q \therefore a = +q \text{ and } b = -q;$$

which reduces Eq. (11) to

$$y = C_1 e^{qx} + C_2 e^{-qx} - \frac{p^2 e^{qx}}{2q}\int xe^{-qx}\,dx + \frac{p^2 e^{-qx}}{2q}\int xe^{qx}\,dx.$$

To integrate $\int xe^{-qx}\,dx$ first differentiate xe^{-qx},

$$\therefore d\,(xe^{-qx}) = -qxe^{-qx}\,dx + e^{-qx}\,dx,$$

from which we find

$$xe^{-qx}\,dx = \frac{1}{q}e^{-qx}\,dx - \frac{1}{q}d\,(xe^{-qx})$$

$$\text{or } \int xe^{-qx}\,dx = \frac{1}{q}\int e^{-qx}\,dx - \frac{1}{q}\int d\,(xe^{-qx})$$

The integral of $\int e^{-qx}\,dx$ is $-\frac{1}{q}e^{-qx}$ and the last term is the integral of the differential of a quantity, hence

$$\int xe^{-qx}\,dx = -\frac{1}{q^2}e^{-qx} - \frac{1}{q}xe^{-qx} = -\frac{1}{q^2}e^{-qx}(1+qx)$$

$$\therefore \frac{p^2e^{qx}}{2q}\int xe^{-qx}\,dx = -\frac{p^2}{2q^3}(1+qx).$$

Similarly the last term becomes $-\frac{p^2}{2q^3}(1-qx)$; and hence the former subtracted from the latter gives $\frac{p^2}{q^2}x$.

$$\therefore y = C_1e^{qx} + C_2e^{-qx} + \frac{p^2}{q^2}x.$$

The following special solution, which is more simple for this case, is communicated by Prof. S. W. Robinson, of the Industrial University of Illinois.

To integrate

$$\frac{d^2y}{dx^2} = q^2y - \rho^2x$$

Differentiate twice and

$$\frac{d^4y}{dx^4} = q^2\frac{d^2y}{dx^2}.$$

Put $\frac{d^2y}{dx^2} = u$ and differentiate twice and $\frac{d^4y}{dx^4} = \frac{d^2u}{dx^2} = q^2\frac{d^2y}{dx^2}$

$$= q^2u = \frac{d^2u}{dx^2} \quad \therefore q^2udu = \frac{dud^2u}{dx^2},$$

$$\text{or } \frac{du^2}{dx^2} = q^2u^2 + C = q^2\left(u^2 + \frac{C}{q^2}\right)$$

$$\text{or } x = \int\frac{du}{q\sqrt{u^2+\frac{C}{q^2}}} = \frac{1}{q}\log\left(u + \sqrt{u^2+\frac{c}{q^2}}\right) - \log C_1,$$

$$\text{or } e^{qx} = \frac{u + \sqrt{u^2+\frac{C}{q^2}}}{C_1}$$

or $C_1 e^{qx} - u = \sqrt{u^2 + \frac{C}{q^2}}$, square, and $C_1^2 e^{2qx} - 2C_1 e^{qx} u + u^2 = u^2 + \frac{C}{q^2}$

$$\therefore u = \frac{C_1 e^{qx}}{2} - \frac{Ce^{-qx}}{2C_1 q^2}$$

$$\text{But } \frac{d^2y}{dx^2} = u = q^2 y - p^2 x = \frac{C_1 e^{qx}}{2} - \frac{Ce^{-qx}}{2C_1 q^2}$$

$$\text{or } y = \frac{C_1}{2q^2} e^{qx} - \frac{C}{2C_1 q^4} e^{-qx} + \frac{p^2}{q^2} x$$

which is the same as in text by putting

$$\frac{C_1}{2q^2} = C_1 \text{ and } \frac{C}{2C_1 q^4} = - C_2$$

Equation (131) gives

$A = 0$; $B = q^2$ $\therefore m = \pm q\sqrt{-1}$ $\therefore a = q\sqrt{-1}$ and $b = -q\sqrt{-1}$ and $X = p^2 (l - x)$.

Make $\Delta - y = -y'$, which in Eq. (11) above gives

$$-y' = C_1 e^{qx\sqrt{-1}} + C_2 e^{-qx\sqrt{-1}} - \frac{p^2}{q^2}(l - x) = \Delta - y.$$

But we have (Chauvenet's *Trigonometry*, p. 128, Eq. (432)),

$$e^{qx\sqrt{-1}} = \cos qx + \sqrt{-1} \sin qx$$

$$e^{-qx\sqrt{-1}} = \cos qx - \sqrt{-1} \sin qx.$$

Make $C_1 + C_2 = A_1$ and $(C_1 + C_2)\sqrt{-1} = B_1$ and we find

$$\Delta - y = A_1 \cos qx + B_1 \sin qx - \frac{p^2}{q^2}(l - x).$$

Let $A_1 = A \sin q B$
$B_1 = A \cos q B$

$$\therefore \Delta - y = A \sin q (x + B) - \frac{p^2}{q^2}(l - x).$$

115. Suppose that the column is fixed at the lower end and has a weight P at the upper end. Take the origin at the lowest end, and let Δ be the deviation of the upper end from a vertical. Then

$$EI \frac{d^2y}{dx^2} = P(\Delta - y).$$

The complete integral gives

$$y = \Delta + A \cos \sqrt{\frac{P}{EI}}\, x + B \sin \sqrt{\frac{P}{EI}}\, x,$$

in which A and B are the constants of integration.

$$y = 0 \text{ for } x = 0 \text{ and } \frac{dy}{dx} = 0 \text{ for } x = 0$$

$$\therefore A = -\Delta \text{ and } B = 0$$

$$\therefore y = \Delta\left(1 - \cos\sqrt{\frac{P}{EI}}\,x\right)$$

$$y = \Delta \text{ for } x = l \;\therefore \cos\sqrt{\frac{P}{EI}}\,l \text{ or}$$

$$\therefore \sqrt{\frac{P}{EI}}\,l = \tfrac{1}{2}(2n + 1)\pi$$

in which n is an integer.

$$\text{If } n = 0$$

$$P = \frac{\pi^2 EI}{4l^2}$$

The general equation of the curve is

$$y = \Delta\left(1 - \cos\frac{2n + 1}{2l}\pi x\right)$$

FIG. 127.

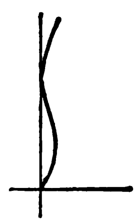

FIG. 128.

If $n = 2$, the curve is represented by Fig. 128.

A solution was given by Prof. Rankine in the *Civ. Eng.* and *Arch. Jour.* for 1863, p. 65, by which the *deflection* may be computed.

146. See Article 97 of Appendix.

To find the inclination so as to give a minimum strength, make the first differential coefficient of Eq. (172) equal zero. This gives

$$0 = (d^3 - 2b^2d)\sin^2 i\cos i + (2bd^2 - b^3)\sin i\cos^2 i + bd^2\sin^3 i - b^2d\cos^3 i.$$

But $\sin^3 i = \sin i\,(1 - \cos^2 i)$ and $\cos^3 i = \cos i\,(1 - \sin^2 i)$ which will reduce the preceding equation to

$$(d^3 - b^2d)\sin^3 i\cos i + (bd^2\ b^3)\sin i\cos^2 i + bd^2\sin i - b^2d\cos i = 0.$$

In this substitute $\sin i = \sqrt{1 - \cos^2 i}$ and reduce, and we obtain

$[-(bd^2 - b^3)^2 - (d^3 - b^2d)^2] \cos^6 i + [(bd^2 - b^3)^2 - 2(bd^3 - b^3)\,bd^2 - 2$ $(2b^2d - d^3)(b^3 - b^2d)] \cos^4 i + [2(bd^2 - b^3)\,bd^2 - b^2d^4 - (2b^2d - d^3)^2]$ $\cos^2 i = -b^2d^4$ ....(1)

Discussion of Equation (1).

1. Let $d = nb$; then we have

$$(n^6 - n^4 - n^2 + 1)\cos^6 i - (2n^6 - 7n^4 + 4n^2 + 1)\cos^4 i + (n^6 - 5n^4 + 6n^2)\cos^2 i = n^4;$$

hence the angle depends only upon the ratio of the sides.

2. Let $n = 1$.
This reduces to $\cos^2 i = \frac{1}{2}$.

$$\therefore \cos i = +\sqrt{\tfrac{1}{2}}. \quad \therefore i = 45° \text{ or } 135°.$$

These values give

$$\frac{1}{6}\frac{Rb^3}{\sqrt{2}} = \frac{1}{6}Rb^3 \times 0.70710 +, \text{ for the strength of the beam.}$$

If $b = d$ and $i = 90°$, we find $\frac{1}{6}Rb^3$ for the strength of the beam.
Hence the strength of a square beam with its side vertical, is to the strength of the same beam with its diagonal vertical as 1 to 0.70710. But if the condition be that the beam shall in both cases be completely severed, then the latter fraction must be multiplied by 1.09125 +, as shown in Article 149. Then the ratio becomes as 1 to 0.77162 +.

3. Let $n = 2$ and $\cos^2 i = y$.
Then the equation becomes

$$y^3 - \tfrac{33}{45}y^2 + \tfrac{8}{45}y = \tfrac{16}{45} \text{..........................(274).}$$

To make the second term disappear, make $y = z + \frac{11}{45}$, and it becomes

$$z^3 - \frac{3}{45^3}z = \frac{31102}{45^3}$$

This solved gives $z = 0.70112$

$$\therefore y = 0.94556 + = \cos^2 i$$
$$\therefore \cos i = \pm 0.9723 +$$
$$\therefore i = 13° 30' \text{ or } 166° 30'$$

making $d = 2b$, and we have for the strength of the beam

when $i = 13° 30'$	;	$\frac{1}{3}Rb^3 \times 0.8295 +$
" $i = 0°$	;	$\frac{1}{3}Rb^3$
" $i = 90°$	;	$\frac{2}{3}Rb^3$.

It is probable that in the inclined position the angle would fracture before the beam is loaded to its ultimate strength, but the investigation for determining it would be tedious and unprofitable. Whether this be the case or not, we *see that the beam is not weakest when it rests on its broad side.*

It appears that the side of the beam may be so inclined as to have the same strength as when it rests on its broad side, and the angle of inclination which will fulfil this condition may be found by making Eq. (1) equal $\frac{1}{6}Rb^2d$.

This done, cos i eliminated, and a reduction made, gives;

$$(d^2 - b^2)^2 \sin^3 i - 2(d^2 - b^2)\, bd \sin^2 i + (3b^2d^2 - b^4) \sin i = 2b^3 d.$$

If $d = nb$, we have

$$(n^2 - 1)^2 \sin^3 i - 2(n^2 - 1)\, n \sin^2 i + (3n^2 - 1) \sin i = 2n.$$

If $n = 2$ we have

$$\sin^3 i - \frac{4}{3} \sin^2 i + \frac{11}{9} \sin i = \frac{4}{9}$$

which solved gives $i = 34° 23'$.

149. The moment of inertia of the trapezoid $DEAB$ in reference to an axis passing through C (Fig. 77) and parallel to IJ, equals the moment of inertia of the triangle ABC, *less* that of CDE about the same axis; or $\frac{1}{3} bd^3 - \frac{1}{3} vw^3$.

According to the *formula of reduction*, the moment of the trapezoid in reference to the axis IJ, equals the moment given above, *less* the area of the trapezoid multiplied by the square of the distance CH; or

$$\tfrac{1}{3}(bd^3 - vw^3) - \tfrac{1}{2}(b+v)\,(d_1 + w)^2$$

To find d_1, we have the statical moment of $ABDE$ equal to the statical moment of ABC, less that of CDE. Take the origin of moments at C, and we have

$$ABED \times (w + d_1) = ABC \times \tfrac{2}{3}d - DCE \times \tfrac{2}{3}w$$

$$\text{or, } \tfrac{1}{2}(b+v)\,(w + d_1) = \tfrac{1}{3}bd^2 - \tfrac{1}{3}vw^2.$$

We also have

$$w = \frac{d}{b}v$$

whence by elimination and reduction we find the expressions in the text.

150. To integrate $r^3 \sin^2 \phi\, d\phi\, dr$ substitute $\frac{1}{2}(1 - \cos^2 \phi)$ for $\sin^2 \phi$, which gives

$$\iint r^3 (1 - \cos 2\phi)\, dr d\phi = \tfrac{1}{8} r^4 \phi - \tfrac{1}{16} r^4 \sin 2\phi.$$

151. If a = semi-transverse axis
b = semi-conjugate axis

The equation of the ellipse is

$$a^2y^2 + b^2x^2 = a^2b^2$$

$$\therefore y^3 = \frac{b^3}{a^3}(b^2 - x^2)^{\frac{3}{2}}$$

$$\therefore I = \int_{-y}^{y} \int_{-a}^{+a} y^2\, dy dx = \frac{2}{3} \int_{-a}^{+a} y^3 dx = \frac{2b^3}{3a^3} \int_{-a}^{+a} (b^2 - x^2)^{\frac{3}{2}} dx,$$

which, by applying formula (B), p. 285 of Courtenay's *Calculus*, becomes

$$\frac{2}{3}\frac{b^3}{a^3}\left[\frac{1}{4}x(a^2 - x^2)^{\frac{3}{2}} + \frac{3}{8}a^2x(a^2 - x^2)^{\frac{1}{2}} + \frac{3}{8}a^4 \int_{-a}^{a} \frac{dx}{\sqrt{(a^2 - x^2)}}\right]$$

which for the limits $x = 0$ and $x = a$ becomes $\frac{1}{4}\pi ab^3$.

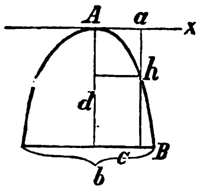

FIG. 129.

152. The origin being at the vertex, x horizontal, the equation for the curve is of the form $x^2 = 2py$; or for the point B, $\frac{1}{4}b^2 = 2pd$ $\therefore$ $2p = \frac{b^2}{4d}$; hence the equation is $x^2 = \frac{b^2}{4d}y$ $\therefore$ $y = \frac{4d}{b^2}x^2$; hence by substituting the value of I and integrating we have

$$\therefore I = \int_{-x}^{x}\int_{0}^{d} y^2 dydx = 2\int_{0}^{d} y^2 xdy = \frac{b}{\sqrt{d}}\int_{0}^{d} y^{\frac{5}{2}}dy = \frac{2}{7}bd^3$$

Or thus:

$I = \iint y^2 dydx = \frac{1}{3}\int y^3 dx$. The part $\frac{1}{3}y^3 dx$ is the moment of ah, but we want the part hc, which is within the parabola; hence the limits for y are $d = ac$ and $y = ah$; and for x; $\frac{1}{2}b$ and $-\frac{1}{2}b$,

$$\therefore \int_{y}^{d}\int_{-\frac{1}{2}b}^{+\frac{1}{2}b} y^2 dydx = \frac{1}{3}\int_{-\frac{1}{2}b}^{+\frac{1}{2}b}(d^3 - y^3)\,dx = \frac{1}{3}\int_{-\frac{1}{2}b}^{+\frac{1}{2}b}\left(d^3 - \frac{64d^3}{b^6}x^6\right)dx = \frac{2}{7}bd^3$$

as before.

To find the moment in reference to the axis passing through the centre and parallel to the base, we may use the formula of reduction and obtain

$$\frac{2}{7}\,bd^3 - \frac{2}{3}\,bd \times \left(\frac{3}{5}\,d\right)^2 = \frac{8}{175}bd^3$$

160. *a.* Expand and we have

$$\delta x\int ydx - \delta\int xydx = \frac{1}{6}Ry^2$$

Consider x as the independent variable, and differentiate twice, observing that the differential of dx will be zero, and the differential of the integral is the original quantity, and we obtain

$$dx\int ydx = \frac{R}{6\delta}\,d\,(y^2), \text{ and}$$

$$\frac{d^2(y^2)}{dx^2} = \frac{6\delta}{R}y$$

Let $y^2 = z$ $\therefore$ $y = z^{\frac{1}{2}}$ and $d^2(y^2) = d^2z$, which, substituted in the preceding, gives

$$\frac{d^2z}{dx^2} = \frac{6\delta}{R}\,z^{\frac{1}{2}}.$$

Multiply by dz and integrate, and we have

$$\frac{dz^2}{dx^2} = \frac{8\delta}{R}\,z^{\frac{3}{2}} + C_1$$

But $\frac{dz}{dx} = 0$ for $y = 0$ or $x = 0$ $\therefore$ $C_1 = 0$.

Hence

$$z^{-\frac{3}{4}}\, dz = \sqrt{\frac{8\delta}{R}}\, dx$$

of which the integral is

$$4z^{\frac{1}{4}} = 4y^{\frac{1}{2}} = \sqrt{\frac{8\delta}{R}}x + C_2$$

But $y = 0$ for $x = 0$ $\therefore$ $C_2 = 0$,

$$\therefore y = \frac{\delta}{2R}x^2.$$

If $\frac{dz}{dx}$ is not zero for $y = 0$, it can be integrated only by Elliptic Functions.

b. By two differentiations we obtain

$$\text{1st, } \delta dx \int u dx = \frac{Rd}{6}\, du.$$

$$\text{2d, } \frac{d^2u}{dx^2} = \frac{6\delta}{Rd}\, u.$$

The first integral is

$$\frac{du^2}{dx^2} = \frac{6\delta}{Rd}\, u^2 + C,$$

$$\therefore dx = \sqrt{\frac{Rd}{6\delta}}\, \frac{du}{\sqrt{\frac{Rd}{6\delta}\, C + u^2}}$$

$$\therefore x = -\sqrt{\frac{Rd}{6\delta}}\, \log_e \left[\sqrt{\frac{Rd}{6\delta}\, C + u^2} - u\right] + C'$$

The curve will be an asymptote to the axis of x, and hence will be of infinite length. If at the origin $u = 1$ for $x = 0$, the value of C' may be determined. Similarly if $\frac{du}{dx} = a$ at the origin C may be determined.

c. By expanding we obtain

$$\delta x. \int y^2 dx - \delta \int y^2 x dx = \tfrac{1}{4} Ry^3.$$

The first differential is

$$\delta dx \int y^2 dx = \tfrac{1}{4} Rd(y^3).$$

The second differential coefficient is

$$\frac{d^2(y^3)}{dx^2} = \frac{4\delta}{R}\, y^2.$$

Let $y^3 = z$ and the integration may be performed as in the preceding cases; the constants of integration being considered zero.

179. Differentiate Equation (213), considering U as constant, and we have

$$G\frac{dg}{dy}=\frac{\Sigma P}{I}y$$

This expression has been generalized. Consider g^1 as the amount of transverse shearing per unit in the direction of U, and G^1 the coefficient for the same. Then we have

$$G\frac{dg}{dy}+G'\frac{dg'}{dy}=\frac{\Sigma P}{I}y$$

Referring g' and g to the co-ordinate axis, and $g=\frac{dy}{dx}\therefore\frac{dg}{dx}=\frac{d^2y}{dx^2}$ and we have

$$G\frac{d^2y}{dx^2}\times G'\frac{d^2z}{dx^2}=\frac{\Sigma P}{I}y$$

On page 216 it is remarked that Eq. (213) is not generally exact. Conceive that the transverse section is an ellipse. Conceive also that it is divided into several vertical strips. If the longitudinal shearing increases from the surface to the neutral axis, then will there be a greater shearing strain on those strips near the middle than on the outer ones. Hence the tranverse sections will not be cylindrical (having for base a curve whose Eq. is (217a)), but they will be warped or generally distorted. This is a refinement, however, which it is not necessary to consider in practice.

On the compressed side of the beam, the quantity $\frac{y^2\Sigma P}{8EI}$ (p. 228) should be subtracted from g in finding the equation of the curve.

190. CRYSTALLIZATION.—Several illustrations of apparently unmistakable crystallization have recently been brought to the attention of the author.

In the process of forging a large shaft for a sea-going steamer, the steam hammer was at work upon one end, as in the case of the "porter bar" already referred to in the text, and while the mass was gradually being reduced in size, a piece of the opposite end broke off at a point where it was 16 inches in diameter. The fracture was partly granular and partly crystalline. One crystal was unmistakably cubic, and its facets were nearly a half-inch square.

At the Washington Navy Yard, recently, the large testing machine, which was constructed for a strain of 300 tons, broke down under a pull of 100 tons. The rod which was broken had been in use 35 years, and, during that period, had been subjected to many heavy strains and violent shocks. Not long before it was broken it had been subjected to a tension of 288,000 pounds. It was 5 inches in diameter, and was originally supposed to have had a resistance of about 1,000,000 pounds. The fracture presents a granular structure, with here and there laminæ composed of crystals. Some of these crystals are large and well defined. The laminæ or strata preserves their characteristic peculiarities, whether of granulation or of crystallization, lying parallel to the axis and extending from the point of original fracture to a section about a foot distant where the bar was broken a second time by a steam hammer. It is thus shown to be the fact that when such true crystallization does occur, it pervades a considerable extent, if not the whole of the piece. It thus differs from the granular structure which distinguishes the surfaces of a fracture suddenly produced, and which is so generally confounded with real crystallization.

The above instance is given by Prof. Thurston, who also describes the following case. A pupil of the Stevens' Institute of Technology, employed in the instrument makers' workshop, in annealing a number of steel hammers, left them exposed to the high temperature of the furnace about twelve hours. When finishing one of them, a careless blow broke it, and the fractured surface was found to have a distinctly crystalline character.

In this example, however, the facets were all pentagonal and were usually very perfectly formed.

"These illustrations," our informant remarks, "are conclusive of the question whether iron may crystallize. When imperfect cubic crystals are developed, it is easy to mistake them, but the formation of pentagonal dodecahedra, in large numbers and in perfectly accurate forms, may be considered unmistakable evidence of the fact that iron may crystallize in the cubic or a modified system. This may apparently take place either by very long continued jarring, or under the action of high temperature, by either mechanical or physical tremor long continued. But no evidence is given here that a single suddenly applied force producing fracture can cause such a systematic and complete rearrangement of molecules. The granular fracture produced by sudden breaking, and the crystalline structure produced as above during long periods of time, are as distinct in nature as they are in their causes.

"But simple tremor, *where no sets of particles are separated so far as to exceed the elastic range*, and to pass beyond the limit of elasticity, does not seem to produce this effect, however.

"In fact, some of the most striking illustrations of the improvement in the quality of wrought-iron with time have occurred where severe jarring and tremor was common. As one example, the first wrought-iron T-rails ever made were laid down on the Camden and Amboy Railroad in 1832. They were then brittle and of decidedly poor quality. In later years these old rails have been taken up and found to be of excellent quality; and when there has arisen a necessity for a supply of unusually good iron, a lot of these rails has sometimes been taken up and sent to the rolling mill to be made into bar iron.

"Here the metal has been subjected for many years to the strains and tremor accompanying the passage of trains without apparent tendency to crystallization, and with evident improvement in its quality. The fact is stated by gentlemen upon whom perfect reliance may be placed. The improvement noticed is supposed to be due to a surface oxidation of the injurious elements originally present in the iron, and to that tendency to uniform diffusion which gradually supplies new portions from the interior, until the metal, by this gradual removal of those elements, becomes, after many years, comparatively pure. Such a process of diffusion occurs, in the inverse direction, when carbon is introduced into steel by the cementation process. Many illustrations of this improvement of metal with age are familiar to every mechanic. Boiler-makers find chisels and drift-pins which are taken from boilers where they may have been left months previously, almost invariably of excellent quality; other mechanics find that tools long lost and again found rusty from exposure to the weather, are apparently of better quality than before; farmers leave their new scythe blades out of doors from one season to another with a resulting benefit which is not all imaginary."

APPENDIX II.

TABLE

Of the Mechanical Properties of the Materials of Construction.

NOTE.—The capitals affixed to the numbers in this table refer to the following authorities:—

B. Barlow. Report of the Commissioners of the Navy, etc.
Be. Bevan.
Bn. Buchanan.
Br. Belidor, Arch. Hydr.
Bru. Brunel.
C. Couch.
Cl. Clark.
D. Darcel, Annales for 1858.
D. W. Daniell and Wheatstone. Report on the stone for the Houses of Parliament.
E. Eads.
F. Fairbairn.
G. Grant.
H. Hodgkinson. Report to the British Association of Science, etc.
Ha. Haswell. Eng. and Mech. Pocket-Book, 1869.
J. Journal of Franklin Institute, vol. XIX., p. 451.
K. Kirwan.
Ki. Kirkeldy.
La. Lamé.
M. Mischembroeck. Introd. ad Phil. Nat. I.
Ma. Mallet.
Mi. Mitis.
Mt. Mushet.
Pa. Colonel Pasley.
R. Roudelet. L'Art de Bâtir, IV.
Ro. Roebling.
Re. Rennie. Phila. Trans., etc.
S. Styffe. On Iron and Steel.
T. Thompson.
Te. Telford.
Tr. Tredgold. Essay on the Strength of Cast Iron.
W. Watson.
Wa. Major Wade.
Wn. Wilkinson.

* Calculated from the experiments of Fairbairn and Hodgkinson.

NAMES OF MATERIALS.	Weight of one cubic ft. in lbs. s	Tenacity per sq. inch in lbs. T.	Crushing Force per square inch in lbs. C.	Modulus of Rupture. R.	Coefficient of Elasticity. E.
METALS.					
Antimony—					
Cast	281.25	1,066 M.			
Bismuth	613.87	3,250 M.			
Brass—					
Cast	525.00	17,968 Re.	10,304 Re.	....	9,170,000
Wire-drawn	534.00	....	....	....	14,230,000
Copper—					
Cast	537.93	19,072	29,272 Re.		
Sheet	549.06	32,184			
Wire-drawn	560.00	61,228			
In Bolts	....	48,000			
Iron.					
Cast Iron.					
Old Park	....	....	....	48,240 T.	18,014,400 T.
Carron, No. 2—					
Cold Blast	441.62	16,683 H.	106,375 H.	38,556 H.	17,270,500 H.*
Hot Blast	440.37	13,505 H.	108,540 H.	37,503 H.	16,085,000 H.*
Carron, No. 3—					
Cold Blast	443.37	14,200 H.	115,442 H.	35,980 F.*	16,246,966 F.
Hot Blast	441.00	17,775 H.	133,440 H.	42,687 F.*	17,873,100 F.

TABLE.—*Continued.*

Names of Materials.	Weight of one cubic ft. in lbs. δ	Tenacity per sq. inch in lbs. T.	Crushing Force per square inch in lbs. C.	Modulus of Rupture. R.	Coefficient of Elasticity. E.
Iron.					
Cast Iron.					
Devon, No. 3—					
Cold Blast	455.93	....	....	36,288 H.*	22,907,700 H.
Hot Blast	451.81	29,107 H.	145,435 H.	43,497 H.*	22,473,650 H.
Buffery, No. 1—					
Cold Blast	442.43	17,466 H.	93,366 H.	37,503 H.*	15,381,200 H.
Hot Blast	437.37	13,434 H.	86,397 H.	35,316 H.*	13,730,500 H.
Coed Talon, No. 2—					
Cold Blast	434.06	18,855 H.	81,770 H.	33,453 F.*	14,313,500 F.
Hot Blast	435.50	16,676 H.	82,739 H.	33,696 H.*	14,322,500 F.
Elsicar, No. 1—					
Cold Blast	439.37	....	....	34,587 F.*	13,981,000 F.
Milton, No. 1—					
Hot Blast	436.00	....	....	29,889 F.*	11,974,500 F.
Muirkirk, No. 1—					
Cold Blast	444.56	....	....	36,693 F.*	14,003,550 F.
Hot Blast	434.56	....	....	33,850 F.*	13,294,490 F.
Morris Stirling's 2d quality	....	25,764	119,000		
Gun Metal—					
American	....	....	14,000 to 34,000 Wa.	....	27,548,000 Wa.
Extra Specimens	595.00	....	45,970 Wa.		
Steel.					
Hammered Cast Steel, from F. Krupp	....	91,000 } 122,000 } S.	....	....	31,359,000 S.
Tempered	....	171,000 S.			
Bessemer Steel, from Högbo, marked 10	....	140,945 S.	....	....	31,819,000 S.
Bessemer Steel, Eng. Mean of four Experiments	485.37	88,415 F.	225,568 F.	....	29,215,000 F.
Naylor, Vickers & Co. Crucible Steel	488.70	108,099 F.	225,568 F.	....	30,278,000 F.
Mushet's Steel—					
Soft	492.50	93,616 F.	....	....	31,901,000 F.
Cast Steel—					
Soft	486.25	120,000			
Not Hardened	....	....	198,944 Wa.		
Mean Temper	....	....	391,985 Wa.		
Razor Tempered	490.00	150,000	....	....	29,000,000
Steel Wire Rope—					
Fine Wire	....	40,000 Ro.			
Chrome Steel	....	195,000			
Wrought Iron.					
English	481.20	57,300 La.			From 22,000,000 to 28,000,000.
In Bars	475.50 / 487.00	57,300 La.			
Hammered	....	67,200 Bru.			
Russian	....	60,480 La.			
Swedish, in bars	....	71,680 R.			
English, in wire 1-10 inch diam.	....	80,000 Te. / 96,000 Te.			
Russian, in wire; diam. 1-20 to 1-30 inch	....	134,000 La. / 203,000 La.			

TABLE.—*Continued.*

NAMES OF MATERIALS.	Weight of one cubic ft. in lbs. δ	Tenacity per sq. inch in lbs. T.	Crushing Force per square inch in lbs. C.	Modulus of Rupture. R.	Coefficient of Elasticity. E.
Wrought Iron.					
Rolled in sheets and cut crosswise	....	40,320 Mi.			From 22,000,000 to 28,000,000.
Cut lengthwise	....	31,360 Mi.			
In chains, oval links, iron ½ in. diam	....	48,160 Br.			
Wire, American	....	73,600 Ha.			
Lake Superior and Iron Mountain Charcoal Bloom	....	90,000 Ha.			
Missouri Iron, bar	....	47,909 J.			
Tennessee, bar, 21 exp	....	52,099 J.			
Salisbury, Ct., 40 exp	....	58,009 J.			
Centre Co., Pa., 15 exp	....	58,400 J.			
Phillipsburgh Wire, Pa.					
Diam. { 0.333 in., 13 exp	....	84,186 J.			
Diam. { 0.190 in., 5 exp	....	73,888 J.			
Diam. { 0.156 in., 5 exp	....	89,162 J.			
Mean of 188 rolled bars	....	57,557 Ki.			
Mean of 167 plates lengthwise	....	50,737 Ki.			
Mean of 160 plates crosswise	....	46,171 Ki.			
Low Moor, bars	....	60,364 Ki.			
Swedish, forged	....	{ 41,000 Ki. { 50,000 Ki.			
Hammered Bessemer Iron, from Högbo	....	....	....	....	32,320,000 S.
Low Moor Rolled Puddled Iron	....	....	....	....	31,976,000 S.
Rolled Iron, Swedish, charcoal heath	....	65,000 S.	....	....	27,000,000 S.
Lead, cast, English	717.45	1,824 Re.			
Lead Wire	705.12	2,581 M.			
Silver, standard	644.50	40,902 M.			
Tin, cast	455.68	5,322 M.	....	....	4,608,000 Tr.
Zinc	439.25	....	....	....	13,680,000 Tr.
STONE—NATURAL AND ARTIFICIAL.					
Granites.					
Aberdeen, blue	164	....	10,914 Re.		
Cornish	166	....	6,356 Re.		
Killincy, very felspathic	....	....	10,780 Wn.		
Mount Sorrell, granite	166	....	12,286 F.		
Sandstones.					
Caithness Pavement	....	....	6,493 Bn.		
Dundee Sandstone	158	....	6,630 Re.		
Derby Grit, a red, friable Sandstone	148	....	3,142 Re.		
Do. from another quarry	156	....	4,345 Re.		
Limestones.					
Limestone, Magnesian (Grafton, Ill.)	....	....	17,000 E.	....	Same as W't. Iron. E.

TABLE.—*Continued.*

Names of Materials.	Weight of one cubic ft. in lbs. S	Tenacity per sq. inch in lbs. T.	Crushing Force per square inch in lbs. C.	Modulus of Rupture. R.	Coefficient of Elasticity. E.
Limestones.					
Limestone, compact	162		7,713 Re.		
Limestone, Kerry, Listowel Quarry, Eng			18,043 Wn.		
Chalk			501 Re.		
Other Stones.					
Alabaster (Oriental), white	170				
Marble, statuary			3,216 Re.	1,062	25,200,000 T.
Do. white Italian, veined	165		9,681 Re.	2,664	
Do. black Galloway	168		9,219 Re.		
Portland Stone (Oolite)	151		3,792 Re.		
Valentia, Kerry (slate stone)			10,943 Wn.		
Green Stone, from Giant's Causeway			17,220 Wn.		
Quartz Rock, Holyhead (across lamination)			25,500 Ma.		
Quartz Rock (parallel to lamination)			14,000 Ma.		
Gravel	120				
Green Moor	158		2,010 Re.		
Artificial Stone.					
Brick, red	135.5	280	808 Re.		
Brick, pale red	130.31	300	562 Re.		
Brick, common			800 to 4,000 Ha.		
Bire Brick, Stourbridge			1,717 Re.		
Brick, Stock			2,177 Ha.		
Bricks set in cement (bricks not very hard)			521 Cl.		
Brick Masonry, common			500 to 800 Ha.		
Cement, Portland, with sand		92 to 284 D.			
Cement, Portland, with no sand		427 to 711			
Cement, Portland			1,000 to 5,900 G.		
Chalk	116.81		334 Re.		
Glass, Plate	153.31	9,420			
Mortar	107	50	120 to 240 Ha.		
Timber.					
Acacia, English	47.37	16,000 Be.		11,202 B.	1,152,000 B.
Alder	50.00	14,186 M.	6,859 H.		
Apple Tree	49.56	19,500 Be.			
Ash, Ordinary state	43.12	17,207 B.	8,683 H.	12,156 B.	1,644,800 B.
Ash, Very dry	55.81		9,363 H.		
Bay Tree	51.37	12,396	7,158 H.		
Bean, Tonquin	67.51			20,886 B.	2,601,600 B.
Beech, Ordinary	53.37	15.784 B.	7,733 H.	9,336 B.	1,353,600 B.
Beech, Very Dry	45.12	17,850 B.	9,363 H.		

TABLE.—*Continued.*

NAMES OF MATERIALS.	Weight of one cubic ft. in lbs. S	Tenacity per sq. inch in lbs. T.	Crushing Force per square inch in lbs. C.	Modulus of Rupture. R.	Coefficient of Elasticity. E.
TIMBER.					
Birch, common	49.50	15,000	4,533 H. 6,402 H.	10,920 B.	1,562,400 B.
Birch, American	40.50	....	11,663 H.	9,624 B.	1,257,600 B.
Box, dry	60.00	19,891 B.	10,299 H.		
Bullet Tree (Berbice)	64.31	....	....	15,636 B.	2,610,600 B.
Cane	25.00	6,300 Be.			
Cedar, Canadian	56.81	11,400 Be.	5,674 H.		
Crab Tree	47.80	....	6,499 H.		
Deal—					
Christiana Middle	43.62	12,400	....	9,864 B.	1,672,000 B.
Norway Spruce	21.25	17,600			
English	29.37	7.000			
Red	....	....	5,748 H.		
White	....	....	6,741 H.		
Elder	43.43	10,220	8,467 H.		
Elm, seasoned	36.75	13.489 M.	10,331 H.	6,078 B.	699,840 B.
Fir—					
New England	34.56	....	....	6,612 B.	2,191,200 B.
Riga	47.06	11.549 to 12,857 B.	5,748 to 6.586 H.	6,648 B. 7,572 B.	1,328,800 869,600 B.
Hazel	53.75	18,000 Be.			
Lance Wood	63.87	24,696			
Larch—					
Green	32.62	10.220 B.	3,201 H.	4,992 B.	897,600 B.
Dry	25.00	8.900 B.	5,568 H.	6,894 B.	1,052,800 B.
Lignum-vitæ	76.25	11.800 M.			
Mahogany, Spanish	50.00	16,500	8,198 H.		
Maple, Norway	49.56	10.584			
Oak—					
English	58.37	17,300 M.	4,684 to 9,509 H.	10,032 B.	1,451,200 B.
Canadian	54.50	10,253	4,231 to 9,509 H.	10,596 B.	2,148,800 B.
Dantzic	47.24	12,780	....	8,742 B.	1,191,200 B.
Adriatic	62.06	....	....	8,298 B.	974.400 B.
African Middle	60.75	....	....	13,566 B.	2,283,200 B.
Pear Tree	41.31	....	7,518 H.		
Pine—					
Pitch	41.25	7,818 M.	....	9,792	1,225,600 B.
Red	41.06	....	5,375 H.	8,946 B.	1,840,000 B.
American Yellow	28.81	....	5,445 H.	....	1,600,000 Tr.
Plum Tree	49.06	11,251	3,657 to 9,367 H.		
Poplar	23.93	7,200	3,107 to 5.124 H.		
Teak, dry	41.06	15.000 B.	12,101 H.	14,722 B.	2,414.400 B.
Walnut	41.93	8,130 M.	6,635 H.	....	306,000
Willow, dry	24.37	14,000 Be.			

2B